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THE MATHEMATICS STUDENT

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TOPICS IN THE THEORY OF ANALYTIC FUNCTIONS(I)

ANALYTIC FUNCTIONS, THEIR REPRESENTATION, ANALYTIC CONTINUATION AND SINGULARITIES.

BY

V. GANAPATHY IYER

Introduction.

This forms the first of a series of articles in which it is proposed to give a short account of the fundamental principles and recent developments in the theory of analytic functions. The reader is supposed to be familiar with the theory of complex numbers, their representation in the complex plane (usually known as Argand's diagram), the notion of limits and convergence of complex sequences and series, the idea of uniform convergence, and the properties of polynomials, rational functions, exponential, trigonometric and logarithmic functions. For the rest, short accounts of new ideas are given wherever these are required.

(1·1) *Some notions regarding point sets.*

For the purpose of illustrating the ideas introduced below we require some simple point sets. We denote by $C(a, < r)$ the points z defined by $|z-a|<r$ which form the interior of the circle with centre at $z=a$ and radius r. We denote by $C(a, r)$ the whole circle $|z-a| \leqslant r$, by $c(a, r)$, the circumference $|z-a|=r$, by $S(0, 1)$ the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ where $z=x+iy$, and by $S_r(0, 1)$ those points of $S(0, 1)$ for which x and y are both rational numbers. Lastly, L denotes a sequence $[z_n]$ such that $\lim_{n\to\infty} z_n = l$, a finite number not equal to any z_n, or $\lim_{n\to\infty} |z_n| = \infty$.

(1·2.) Let, now, S be a point set in the complex plane, that is, an aggregate of complex numbers the individual members of which can be determined by some specified law or can be characterised

by some specific property. A point z_0 of S is said to be an *isolated point* of S if there exists a C (z_0, r) which contains no other point of S than z_0. All points of the set L defined in § 1·1 are isolated points of L. A point z_0, *belonging to* S *or not*, is said to be a *limit point* of S if *any* C (z_0, r) contains an infinity of points of S. The point $z = l$ or ∞ is a limit point of L not belonging to L. All points of each of the remaining sets defined in § 1·1 are limit points of the respective sets. The set of all limit points of S is called its *first derived set* and is usually denoted by S′. For instance, C′ $(a, <r) \equiv$ C′ $(a, r) \equiv$ C(a, r) while S'_r $(0, 1) \equiv$ S′$(0, 1) \equiv$ S $(0, 1)$. A set S is said to be *closed* if all its limit points belong to the set. The sets C (a, r), c (a, r) and S $(0, 1)$ are closed while the rest of the sets defined in § 1·1 are not closed. A set S is said to be *dense in itself* if every point of S is a limit point of S_2. All the sets of § 1·1 except L are dense in themselves. A set S is said to be *perfect* if it is closed as well as dense in itself. The sets C (a, r), c (a, r) and S $(0, 1)$ are perfect; the rest are not perfect.

(1·3.) Let S_1 and S_2 be two point sets. We denote by $S_1 + S_2$ the point set consisting of the totality of the points of S_1 and S_2. The set of points common to S_1 and S_2 is denoted by $S_1 . S_2$. We denote by $S_1 - S_2$ the points of S_1 *not* belonging to S_2. For instance, C $(a, <r) + c(a, r) \equiv$ C(a, r); C $(a, <r)$. C $(a, r) \equiv$ C $(a, <r)$; C (a, r). c $(a, r) \equiv c$ (a, r); C $(a, <r)$. c $(a, r) \equiv 0$ (the nul set); and C $(a, r) - c$ (a, r) = C $(a, <r)$. If S_1 is contained in S_2, $S_2 - S_1$ is called the complement of S_1 with respect to S_2. The set S_1 is said to be *everywhere dense* in S_2 (or with respect to S_2) if every point of S_2 is either a point of S_1 or a limit point of S_1. The set S_r $(0, 1)$ is everywhere dense in S $(0, 1)$ and also in S $(0, 1) - S_r$ $(0, 1)$.

(1·4.) A point set S is said to be *bounded or limited* if all points of S lie inside some circle of finite radius. Let S be a bounded set. The points of the complex plane fall into three classes with respect to the set S. A point z_0 is said to be an *interior point* of S if there is a C(z_0, r) every point of which belongs to S. A point z_0 is said to be an *exterior point* of S if there is a C (z_0, r) no point of which belongs to S. A point z_0 which is neither an interior nor an exterior point is said to be a *Frontier* point of S. The set of all Frontier points of S is denoted by F{S}; it is always a closed set, a set consisting of a *finite number of points* being always considered to be closed. Every point of C $(a, <r)$ is an interior point; every point z defined by $|z - a| > r$ is an exterior point of C $(a, <r)$ while F{C$(a, <r)$} $\equiv c(a, r)$. The set S_r $(0, 1)$ has no interior points; all its points are frontier points; F{S_r $(0, 1)$} $\equiv$ S$(0, 1)$ while any point outside S $(0, 1)$ is an exterior point of S_r $(0, 1)$.

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A set S is said to be *open* if all its points are interior points. The set C $(a, < r)$ is open. A set S is said to be *connected* if any two of its points can be connected by a broken line (a polygonal line) all points of which belong to S. An open connected set is said to be a *domain*. And it is with domains we shall be chiefly concerned in the sequel. The set C $(a, < r)$ is a domain; the set defined by $1 < |z| < 2$ (a ring) is a domain.

(1·5.) *Infinity in the complex plane and unbounded sets.*

In § 1·4 we have supposed the set S to be bounded. If S is not bounded we call it unbounded or unlimited. To deal with unbounded sets we have to consider the nature of infinity in the complex plane. We are familiar with the idea of a line at infinity in a plane in projective geometry; that is, in the projective plane we regard infinity as composed of points lying on a straight line. But *in the complex plane we regard infinity as a single point.* In inversion geometry also we consider infinity as a single point. To discuss the nature of the point $z = \infty$ with respect to any set S, we choose a finite number α and set $z' = 1/(z-\alpha)$. The point set $S = S(z)$ will become another $S\alpha = S\alpha('z)$ in the z' – plane. *Any property of $z' = \alpha$ with respect to S_α is regarded as the corresponding property of $z = \infty$ with respect to S.* For instance, the point $z = \infty$ is an interior point of the set S if all points defined by $|z - a| > r$ belongs to S for a properly chosen r.

(1·6.) *A simple Jordan curve* is the set of points $z = x + iy$ defined by

$$\left.\begin{array}{l} x = u(t) \\ y = v(t) \end{array}\right\}, \quad 0 \leqslant t \leqslant 1, \qquad \ldots \quad (1)$$

where $u(t)$ and $v(t)$ are continuous functions of t in $0 \leqslant t \leqslant 1$ and are such that the equalities $u(t_1) = u(t_2)$, $v(t_1) = v(t_2)$ do not *simultaneously hold* for $0 \leqslant t_1 \leqslant 1$, $0 \leqslant t_2 \leqslant 1$ except when $t_1 = t_2$; that is, no two distinct values of t give rise to the same point (x, y). If $u(t)$, $v(t)$ satisfy the above conditions except that $u(0) = u(1)$, $v(0) = v(1)$, the curve is said to be closed. If the length of the curve defined by (1) is finite, we say that the curve is rectifiable.* We shall mean by a *contour* a simple closed rectifiable Jordan curve. A simple closed Jordan curve divides the plane into two mutually exclusive parts. One, a bounded point set, is called its *interior*. The other, an unbounded one, is called its *exterior*. Both the interior and exterior are domains; if a point of one be joined to a point of other by simple Jordan curve or arc, the latter has

* For the definition of the length of a curve, see E. Goursat *Cours d' Analyse* Vol. I, the chapter dealing with curves and surfaces; or any work on differential geometry.

at least one point in common with the closed curve. The closed curve itself forms the set of all Frontier points of both the interior and the exterior of the curve. We may note, in passing, that a broken (or polygonal) line joining any two points is a simple rectifiable Jordan curve, the line being supposed to have no double points.†

(1·7.) *Connectivity of domains.*‡ Let D be a domain.

The set F{D} is called its boundary (§1·4) and will be denoted by B(D). We mean by a *cross-cut* of D a simple Jordan curve joining two points α and β of B(D) such that all its points except α and β are points of D. We say that D is *simply connected* if *any* cross-cut separates D into two distinct pieces, that is, a point of one piece cannot be joined to a point of the other without encountering a point of the cross-cut. The set $C(a, < r)$, the points of S (0, 1) not lying on its sides, and the interior of an ellipse are examples of simply connected domains. A simply connected domain has the further property that any simple closed Jordan curve lying in D can be deformed so as to reduce to a single point without meeting any point of B(D) in the process of deformation. The connectivity of a domain D is said to be p if we *can find* $(p-1)$ cross-cuts which do not render D into two distinct pieces; and any p^{th} cross-cut will separate D into two distinct pieces; in this process we add each cross-cut to the boundary of D before making the next cross-cut. The ring $1 < |z| < 2$ is of connectivity 2 or doubly connected. Let us draw p small circles all outside one another and lying in $C(a, < r)$. The domain D_p inside $C(a, < r)$ but outside the p small circles is p-ply connected.

2. *Analytic Functions.*

There are two points of view from which the theory of analytic functions can be developed. We shall, first, give an account of the one due two Cauchy. We shall discuss later the other method due to Weierstrass and point out the relations between the two towards the end of this article.

(2·1.) Let $f(z)$ be a function defined at every point of a set S in the complex plane. It is said to be single-valued on S if to each z in S corresponds one and only one value of $f(z)$. Otherwise it is

† For a good account of the ideas contained in § 1·1 – § 1·6, see Hobson, *Theory of Functions of a real variable*, Vol. I.

‡ The notion of connectivity becomes important in connection with conformal representation. See P. Dienes, *Taylor Series*, the chapters on geometrical language and conformal representation; also L. R. Ford, *Automorphic Functions*, the chapters on uniformisation.

said to be multiple-valued or many-valued. A proper treatment of multiple-valued analytic functions is possible only after we have discussed the Weierstrassian point of view of analytic functions. Therefore, at present, we confine ourselves to single-valued functions.

(2·2.) Set $f(z)$ be a single-valued function defined on a bounded set S. Let z_0 be a non-isolated point of S. The function $f(z)$ is said to be *analytic at* z_0 *with respect to* S if

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \qquad \ldots \quad (2)$$

exists as a finite number, h being such that z_0+h belongs to S. If z_0 be an isolated point of S we *agree to regard* $f(z)$ *as analytic at* z_0. If $f(z)$ be analytic at every point of S we say that it is *analytic on* S (*or in* S). It is important to note that the point set S (which might be called the domain of definition) on which the analyticity of $f(z)$ is to be considered is as essential an element in the above definition as the existence of the limit (2). The limit (2) has been *defined only with respect to the points of S.* For instance, consider the function

$$f(z)=1, \quad z \text{ on } S_r\ (0, 1)\ [\S\ 1{\cdot}1]. \qquad \ldots \quad (3)$$

the function (3) is analytic on S_r $(o, 1)$ according to the above definition. But if

$$f(z)= \left.\begin{array}{l} 1, \ z \text{ on } S_r\ (0, 1) \\ 0, \ z \text{ on } S\ (0.\ 1)-S_r\,(0, 1) \end{array}\right\} \qquad \ldots \quad (4)$$

$f(z)$ cannot be analytic at any point of S $(o, 1)$ on which it is defined even though the values of $f(z)$ on S_r $(0, 1)$ in (3) and (4) are indentical. Hence we see that analyticity cannot be dissociated from the domain of definition of the function since the existence of limit (2) is considered only with respect to the domain of definition.

(2·3.) *Unbounded sets.* In § 2·2 we have supposed S to be bounded. If S be unbounded we resort to the procedure mentioned in § 1·5. We can conveniently choose $\alpha=o$ in § 1·5 and say that any property of $f(z)$ at $z=\infty$ is interpreted as the corresponding property of $f(z')=f\left(\frac{1}{z}\right)$, $z'=\frac{1}{z}$ at $z'=0$. For instance the function $f(z)=\frac{1}{z}$ is analytic at infinity as well as at all points of the complex plane except at $z=0$. When we speak of the complex plane we include $z=\infty$ as well. When we exclude $z=\infty$ from the complex plane we shall term it *the finite plane.*

3. *Chief properties of single-valued analytic functions.*

The point set S considered in the definition of an analytic function in § 2·2 is quite arbitrary. Very few significant properties

of analytic functions can be deduced if we keep to such general sets. For instance, *any* single-valued function defined at the points of the set L of § 1·1 is analytic according to the definition of § 2·2. So we confine ourselves to open sets. Let $f(z) = P(x, y) + i\,Q(x, y)$, $z = x + iy$, be a single-valued function analytic on an open set S. Then the existence of the limit (2) involves the relations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \qquad \ldots \quad (5)$$

for any point $z = x + iy$ of S. The relations (5) are known as Cauchy-Riemann equations. Conversely, if $P(x, y)$ and $Q(x, y)$ are functions satisfying (5) and the first partial derivatives involved are continuous,* then the limit (2) exists for $f(z) = P(x, y) + i\,Q(x, y)$ for z in S and $f(z)$ is therefore analytic on S. From (5) we deduce that P and Q are solutions of the Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \qquad \ldots \quad (6)$$

If a solution P of (6) be given, we can determine Q from (5) in such a manner that $f(z) = P + i\,Q$ is analytic on S and Q is unique but for an additive constant.

(3·1.) *Integration on curves.* Let C be a simple rectifiable Jordan curve whose end points are $z = a$ and $z = b$. Let $a, z_1, z_2, \ldots, z_\gamma, b$ be points on C in the order of increasing arc measured from $z = a$. Let $f(z)$ be a function continuous on C. Then the sum

$$\sum_{p=o}^{\nu} (z_{p+1} - z_p)\, f(\xi_p)$$

where $z_o = a$, $z_{\nu+1} = b$ and ξ_p is a point on the arc (z_{p+1}, z_p) of C, tends to a definite limit as $d_\nu = \max_{(p)} |z_{p+1} - z_p|$, $p = 0, 1, \ldots \nu$, tends to zero, this limit being independent of the choice of ξ_p on the arc (z_{p+1}, z_p). This limit is called the integral of $f(z)$ along C and is denoted by

$$\int_C f(z)\, dz. \qquad \ldots \quad (7)$$

It must be noted that there is a definite direction, namely, that of increasing arc measured from a, along the curve C implied in the definition of the integral (7); if this direction were reversed the value of the integral will be the negative of (7).

(3.2.) *Continuous boundary values of analytic functions.* Let S be an open set. We might call F {S} (§ 1·4) the boundary of S as in the

* It is not necessary to suppose the continuity of the partial derivatives (5) for the existence of the limit (2). For a detailed discussion of this point, see *Conditions de monogeneté* by D. Menchoff, Actualités Scientifiques et Industrielles, No. 329.

case of domains (§ 1·7). Let z_0 be a point of F {S} which must be a limit point of S since the latter is an open set. Let $f(z)$ be analytic on S and let $f(z) \to \sigma(z_0)$ as $z \to z_0$ remaining in S. Let this hold for each z_0 in F {S} and let $\sigma(z_0)$ be continuous* as z_0 varies on F {S}. Then we say that $f(z)$ is *analytic on S and continuous on the boundary.* If $f(z)$ be analytic on S + F(S), it is, *a fortiori*, analytic on S and continuous on the boundary since the existence of the limit (2) involves the continuity of $f(z)$ at z_0 with respect to S + F(S).

(3.3.) *Cauchy's integral theorem.* Let C be a contour, I (C) being its interior. Let $f(z)$ be analytic on I (C) and continuous on C. Then

$$\int_C f(z)\, dz = 0. \qquad \ldots \quad (8)$$

The result embodied in (8) is one of the most important properties of analytic functions. If $f(z)$ be analytic on I (C) + C, the relation (8) holds *a fortiori*.

(3.4.) *Direction of integration along contours.* Let C be a contour forming part of the boundary of a domain D. For the purpose of generalising (8) we must have some convention regarding the direction of integration along the contour. Suppose a point describes the contour in such a manner as to leave D to its left. Then we shall agree to regard the direction of motion of such a point as *the positive direction along the contour with respect to D.* The reverse direction is the negative direction with respect to D. For instance, the anti-clockwise direction is the positive direction along $c(a, r)$ with respect to the domain C $(a, < r)$ (*see* § 1·1.) If D is the ring $1 < |z| < 2$, the clockwise direction along $c(0, 1)$ and the anti-clockwise direction along $c(0, 2)$ are the positive direction with respect to D.

(3.5.) We can now state a generalisation of (8). Let D be a domain whose boundary consists of a finite number $C_1, C_2, \ldots C_p$ of contours. Let $f(z)$ be analytic on D and continuous on the boundary. Then

$$\sum_{v=1}^{p} \int_{C_v} f(z)\, dz = 0, \qquad \ldots \quad (9)$$

all the integrals being taken in the positive direction (or, in the negative direction) with respect to D along the respective contours. The result certainly holds if $f(z)$ be analytic on the set $D + C_1 + C_2 + \cdots + C_p$.

* A function $f(z)$ is continuous at $z = z_0$ of the set S if $f(z_0 + h) - f(z_0) \to o$ as $h \to o$, $z_0 + h$ being such that it belongs to S. If $f(z)$ be continuous at all points of S we say $f(z)$ is continuous on S (or in S)

(3.6.) Let D, C_1, C_2, . . , C_p and $f(x)$ be as in § 3·5· Let x be a point of D, then

$$f(x)=\sum_{\nu=1}^{p}\frac{1}{2\pi i}\int_{C\nu}\frac{f(z)}{z-x}\,dz. \qquad \ldots \quad (10)$$

In particular, if $f(z)$ be analytic on the interior I (C) of a contour C and continuous on C, Then

$$f(x)=\frac{1}{2\pi i}\int_{C}\frac{f(z)}{z-x}\,dz, \qquad \ldots \quad (11)$$

for all x in I (C). The formulae (10) and (11) give the value of $f(z)$ in the respective domains in terms of its boundary values. *The value of the integral* (9) *for x exterior to I* (C) *is zero.*

4. *Power Series.*

We denote by $P_a(z)$ or simply P_a an expression of the form

$$\sum_{n=0}^{\infty} a_n\,(z-a)^n. \qquad \ldots \quad (12)$$

There exists a number ρ, $0 \leqslant \rho \leqslant \infty$, such that (12) converges absolutely for $|z-a| < \rho$ and uniformly for $|z-a| \leqslant \rho' < \rho$ while it is non-convergent for $|z-a| > \rho$. If $\rho=0$, (10) converges only at the single point $z=a$. We exclude this case and suppose $\rho>0$. The expression (12) is called a *power series,* ρ its *radius of convergence,** $z=a$ the *centre* of the power series, and $C(a, <\rho)$ its *circle of convergence.* We shall use the notation $c(P_a)$ to denote the circle of convergence of (12) and $c(P_a)$ to denote the circumference $|z-a|=\rho$. It may be noted that a power series is a single-valued analytic function on the circle $C(P_a)$.

(4·1) *Taylor Series.* Let $f(z)$ be single-valued and analytic in $C(a, <r)$. Then

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n, \ldots \qquad \ldots \quad (13)$$

for all z in $C(a, <r)$ so that the radius of convergence of (13) is at least r. The series (13) is called the Taylor series of $f(z)$ in $C(a, <r)$; conversely, a power series (12) represents a single-valued analytic function $f(z)$ in its circle of convergence and

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

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* The radius of convergence ρ is given by

$$\frac{1}{\rho}=\lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

(4·2) *Laurent's series.* Let $f(z)$ be single valued and analytic in the ring $r_1 < |z| < r_2$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} a_{-n} (z-a)^{-n}, \qquad \ldots \quad (14)$$

for all z in the ring $r_1 < |z| < r_2$. The part

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

converges for $|z-a| < r_2$ while the remaining part in (14) converges for $|z-a| > r_1$. In particular, *if* $f(z)$ *be analytic in* C$(a, < r)$ *except may be at* $z = a$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} a_{-n} (z-a)^{-n}, \qquad \ldots \quad (15)$$

for *all* z in C$(a, < r)$ except at $z = a$. The expansions (14) and (15) are called the Laurent series of $f(z)$ in the respective domains.

5. *Regular points, poles and isolated essential singularities.*

A function $f(z)$ expressible as a power series in *some* circle C$(a, < r)$ round $z = a$ is said to be regular at a. This involves that $f(z)$ should be analytic at all points of C$(a, < r)$; conversely, if $f(z)$ be analytic on C$(a, < r)$ it is regular at $z = a$ (§4·1) and *also at all points* of C$(a, < r)$ since the latter is an open set. Hence it follows that if $f(z)$ be regular at $z = a$ there is an r such that $f(z)$ is regular at all points of C$(a, < r)$. Therefore the set of all regular points of a function $f(z)$ contains only *interior* points and so forms an *open set* (§1·4).

(5·1) Suppose, now, that $f(z)$ is a function for which an expansion of the form (15) holds for all z in some C$(a, < r)$ except may be at $z = a$. Suppose, further, that there are only a finite number of negative powers of $(z-a)$ in (15) and that m is the highest index of $(z-a)^{-1}$ in (12). Then we say that $f(z)$ *has a pole of order* m *at* $z = a$. It is to be noted that a pole is *surrounded* by a circle C$(a, < r)$ at all points of which, except $z = a$, $f(z)$ is analytic and hence, regular (§5). Therefore there is no other pole in some circle round the pole $z = a$; that is, the set of poles of a function contains only isolated points. If $f(z)$ be a regular at $z = a$ and $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$ while $f^{(m)}(a) \neq 0$ we shall say that $f(z)$ has *a zero of order* m *at* $z = a$. It is easy to see that the two statements '$f(z)$ has a zero of order m

at $z=a$' and '$1/f(z)$ has a pole of order m at $z=a$' are entirely equivalent. From this we see that

$$\lim_{z \to a} (z-a)^m f(z) = l \neq 0$$

at a pole $z=a$ of order m and conversely. Also $|f(z)| \to \infty$ as $z \to a$ at a pole $z=a$ of $f(z)$.

(5·2) If $f(z)$ has an expression of the form (15) for all z in some $C(a, < r)$ except may be at $z=a$ and there are an infinity of negative powers of $(z-a)$ with non vanishing coefficients we shall say that $f(z)$ has *an isolated essential singularity* at $z=a$. Here, again, we note that if $z=a$ be an isolated essential singularity, there is a $C(a, < r)$ at all points of which except $z=a$, $f(z)$ is analytic and, therefore, regular. Conversely, if $f(z)$ be single-valued and analytic in some $C(a, < r)$ except at $z=a$ (and therefore also regular in $C(a, < r)$ except at $z=a$), the point $z=a$ is either (i) itself a regular point, (ii) a pole, or (iii) an isolated essential singularity and these are mutually exclusive possibilities.

(5·3.) It is important to note that a regular point, a pole or an isolated essential singular point of a function is defined not by the behaviour of $f(z)$ at $z=a$ but by its behaviour at all points of some circle with centre $z=a$, that is, at all points of some *neighbourhood* of a. A regular point, a pole, or an isolated essential singular point $z=a$ is surrounded by some circle $C(a, < r)$ all points of which except, perhaps, $z=a$, are regular points for the function. Hence these may be called neighbourhood properties in contrast to what may be termed point properties. For instance, analyticity of a function is a point property, that is, $f(z)$ can be analytic at $z=z_0$ without being analytic in the neighbourhood of z_0. As an example, consider $f(z)=(x-iy)^2$, $z=x+iy$; here $f'(0)=0$ and so $f(z)$ is analytic at $z=0$; but $f(z)$ is not analytic at any point $z \neq 0$ and therefore $f(z)$ is not regular at $z=0$.

(5·4.) Suppose that $f(z)$ is regular * in some $C(a, < r)$ except perhaps at $z=a$. Then an expansion of the form (15) holds for all z in $C(a, < r)$ except at $z=a$. The Coefficient a_{-1} of $(z-a)^{-1}$ in (15) is called the *residue* of $f(z)$ with respect to $z=a$; when $f(z)$ is regular at $z=a$ as well, $a_{-1}=0$ always. Let, now, C be a contour lying in $C(a, < r)$ and containing $z=a$ in its interior. Then

$$\frac{1}{2\pi i} \int_C f(z)\, dz = a_{-1}. \qquad \ldots \quad (16)$$

* It will be sufficiently evident from the previous explanation that the regularity of a single-valued function and its analyticity are *equivalent in an open set.*

(5.5.) *The theorem of residues.* Let D be a domain bounded by a finite number $C_1, C_2, \ldots, C_p$ of contours so that the boundary $B(D) \equiv C_1 + C_2 + \ldots + C_p$. Let $f(z)$ be analytic on D except at a finite number $a_1, a_2, \ldots, a_n$ of points of D. Let $f(z)$ be continuous on B (D).

Then

$$\sum_{v=1}^{p} \frac{1}{2\pi i} \int_{C_v} f(z)\, dz = \sum_{v=1}^{n} R(a_v), \ldots \qquad \ldots \quad (17)$$

where $R(a_v)$ is the residue of $f(z)$ at $z = a_v$ and the integral along C_v ($v = 1, 2, \ldots, p$) is taken in the positive direction with respect to D. The relation embodied in (17) is the famous theorem of residues. It is one of the cardinal results of the theory of analytic functions and its applicability * is co-extensive with that of the theory itself.

(5.6) *Analytic extension or continuation.* Let $f_1(z)$ and $f_2(z)$ be two single-valued analytic functions defined on the sets S_1 and S_2 respectively. Let $f_1(z) = f_2(z)$ at the points, if any, of $S_1 \cdot S_2$. Let the function $f(z)$ defined on $S = S_1 + S_2$ by the relations

$$f(z) = \left.\begin{matrix} f_1(z), & z \text{ on } S_1 \\ f_2(z), & z \text{ on } S_2 \end{matrix}\right\} \qquad \ldots \quad (18)$$

be analytic on S†, there being no contradiction in the definition (15) in virtue of the hypothesis at the points of $S_1 \cdot S_2$. Then we say that $f(z)$ can be extended or continued analytically from the set S_1 to the set S_2 and in this case we *agree* to regard both $f_1(z)$ and $f_2(z)$ as a single analytic function defined on S by the relation (18)‡. We shall again meet with this idea in connection with Weierstrass's definition of analytic functions.

6. *Representation of analytic functions.*

One of the main problems in the theory of analytic functions is to obtain an explicit expression or representation of a function $f(z)$

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* For an account and various applications of the theorem of residues, the reader may consult the excellent book *Calcul des Resides,* by E. Lindelof.

† It is not always true that $f(z)$ defined by (18) is analytic on $S = S_1 + S_2$ when $f_1(z)$ is analytic on S_1 and $f_2(z)$ on S_2; see for instance the example defined by (4).

‡ Usually the term analytically extensible or continuable is used only when S_1 and, S_2 have common points. But, theoretically, there is nothing to prevent us from considering the case when S_1 and S_2 have no common points. In certain cases we can say *a priori* that $f(z)$ defined by (18) is analytic on $S_1 + S_2$ when $f_1(z)$ is analytic on S_1 and $f_2(z)$ on S_2. Two important cases are (i) when S_1 and S_2 are open sets and (ii) when when $S_1 + F\{S_1\}$ and $S_2 + F\{S_2\}$ have no common points. In general, the points of $S = S_1 + S_2$ where $f(z)$ given by (18) ceases to be analytic fall into two classes: (i) a point of $S_2 - S_1$ (or $S_1 - S_2$) which is a limit point of S_1 (or S_2); (ii) points of $S_1 \cdot S_2$ which are *not* the limit points of a sequenee $[z_n]$ such that z_n belongs to $S_1 \cdot S_2$, $n = 1, 2, \ldots$ In class ii) we can omit such points as are isolated for either S_1 or S_2.

known to be analytic on a point set S. The favourite method is to determine a sequence $\{f_n(z)\}$ of functions with simple properties (usually polynomials or rational functions) which converge uniformly to $f(z)$ on S. When S is a circle, the Taylor series (13) gives an expression for a function $f(z)$ single-valued and analytic on S in terms of the values of $f(z)$ and its derivation at the centre. The sum to $(n+1)$ terms of (13) is a *polynomial* of degree n which converges to $f(z)$ uniformly on any closed set interior to the circle. More generally, when $f(z)$ is analytic on an open set S, it is possible to find a sequence $R_n(z)$ of rational functions (whose poles can be chosen with a certain degree of arbitrariness) such that

$$\lim_{n\to\infty} R_n(z) = f(z) \qquad \dots (19)$$

on any closed set interior to S. For instance, if $f(z)$ be analytic in the ring $r_1 < |z| < r_2$, the partial sums

$$R_n(z) = \sum_{\nu=1}^{n} \{a_\nu (z-a)^\nu + a_{-\nu}(z-a)^{-\nu}\} \qquad \dots (20)$$

of the Laurent expansion (14) of $f(z)$ gives a sequence of rational functions with poles at $z=a$ satisfying (19) uniformly on any closed set lying in the ring. When S consists of a bounded simply connected domain, or a finite number $D_1, D_2 \dots D_p$ of bounded simply connected domains such that no two of the closed sets $D_\nu + B(D_\nu)$, $\nu = 1, 2, \dots, p$, have a point in common, it is possible to replace the rational functions $R_n(z)$ in (19) by Polynomials $P_n(z)$. For instance, if

$$P_n(z) = 1 + 2(z^2-1) + 2^2(z^2-1)^2 + \cdots + 2^n(z^2-1)^n, \qquad \dots (21)$$

we find that

$$\lim_{n\to\infty} P_n(z) = \frac{1}{1-2(z^2-1)} = \frac{1}{2-3z^2} = f(z), \qquad \dots (22)$$

uniformly on any closed set in the open set defined by

$$|z^2 - 1| < \tfrac{1}{2}, \qquad \dots (23)$$

which consists of two domains D_1 and D_2 * bounded by the two branches C_1 and C_2 of the curve

$$|z^2-1| = 1/2. \qquad \dots (24)$$

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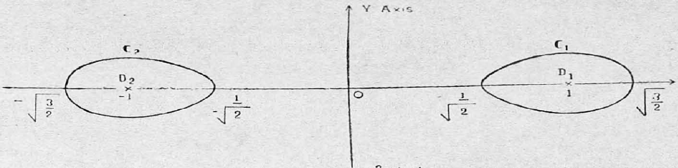
* The function $f(z) = 1/(3-2z^2)$ is analytic in a simply connected domain containing the two domains D_1 and D_2 in which (22) is valid. Even if $f(z)$ be arbitrarily defined on D_1 and D_2 so as to be analytic on $D_1 + D_2$ [e.g. $f(z) = z$ in D_1 and $f(z) = z^2$ on D_2] it is possible to find a polynomial sequence $P_n(z)$ tending to $f(z)$ uniformly on any closed set in $D_1 + D_2$. For various questions connected with the representation of analytic functions and proofs of the various results stated in § 6, the reader may consult the recent book '*Interpolation and approximation*', by J. L. Walsh, American Mathematical Colloquium Publication (1935).

An idea of the curve (24) is given in the following figure.



THE CURVE $|z^2-1| = \frac{1}{2}$

It consists of two non-overleaping domains D_1 and D_2 bounded by two oval curves C_1 and C_2. The curve C_1 cuts the real axis in $z = +1/\sqrt{2}$ and $z = +\sqrt{\frac{3}{2}}$ respectively while C_2 cuts it at the corresponding negative points. It may be noted that the relation (22) is not valid for any point outside these two domains.

7. *Weierstrass definition of analytic functions.*

Weierstrass starts from a power series a short account of the properties of which has been given in § 4. We have already mentioned that a power series is a single valued analytic function, according to Cauchy, within its circle of convergence. Here the power series is made the basis of the definition of an analytic function. We shall now give an account of how this is done.

(7·1) Let* $P_a(z)$ be the power series defined by

$$P_a(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n. \qquad \ldots (25)$$

Let b be a point interior to the circle of convergence $C(P_a)$ of $P_a(z)$. Let

$$P_b(z) = \sum_{n=0}^{\infty} \frac{P_a^{(n)}(b)}{n\,!} \, (z-b)^n. \qquad \ldots (26)$$

By using well-known results on the rearrangement of absolutely convergent double series it is easily seen that (26) converges in the circle $|z-b| < \rho - |b|$ where ρ is the radius of convergence of (25) and that $\qquad P_a(z) = P_b(z) \qquad \ldots (27)$

* The definitions and notations of § 4 are used throughout what follows.

for all z in $|z-b|<\rho-|b|$. We call the power series (26) an *immediate continuation* of (25). The radius of convergence of (26) is at least $\rho-|b|$ but it can well be greater. For instance, consider

$$P_0(z)=1+z+z^2+\cdots=1/(1-z). \qquad \dots (28)$$

which is valid in C (0, 1). Let us form $P_{i/2}\,(z)$ by the rule (26). We get

$$P_{i/2}\quad(z)=\sum_{n=0}^{\infty}\frac{(z-i/2)^n}{(1-i/2)^{n+1}}, \qquad \dots (29)$$

whose radius of convergence is $|1-i/2|=\sqrt{5}/2$ which is greater than $1-|i/2|=\frac{1}{2}$

(7.2.) Now, let $P_a(z)$ and $P_b(z)$ be any two power series. We say that $P_b(z)$ is a *continuation* of $P_a(z)$ if there exists a *finite number* $P_{z_1}(z)$, $P_{z_2}(z)$, . . . , P_{z_n}, of power series such that *any* power series (excepting P_a) of the ordered sequence (P_a, P_{z_1}, $\cdots$, P_{z_2}, P_b) is an immediate continuation of the one which *immediately precedes it.* It can be shown that if *P_b is a continuation of P_a then P_a is a continuation of P_b.* To see this it is sufficient to prove that if P_b is an immediate continuation of P_a then P_a is a continuation of P_b and this is easily shown by noting that the radius of convergence of P_b is not less than $\rho-|b|$ which is positive and by taking new centres on the segment joining a and b. Hence the process of continuation is a *reversible* process.

(7.3.) Let $P_a(z)$ be a power series and consider the *totality* of all those power series which are continuations of $P_a(z)$ including $P_a(z)$ itself.* According to Weierstrass, this totality of power series arising from the continuation of a given power series *constitutes an analytic function.* Each power series of this totality is called an *element* of the function. In virtue of the property of reversibility (§ 7·2) we can start from *any* element of a function to obtain all the power series constituting the function. That is, *an element of a function entirely determines it.*

(7.4.) *Continuation along paths, accessible points and singularities.* We shall mean by a *path* $L(a,b)$ joining $z=a$ and $z=b$, a curve defined by

$$\left.\begin{array}{l}x=u(t)\\ y=v(t)\end{array}\right\}\ 0\leqslant t\leqslant 1, \qquad \dots (30)$$

where $z=x+iy$ and $u(t)$ and $v(t)$ satisfy the following conditions:

(i) $u(0)+iv(0)=a,\ u(1)+iv(1)=b$;

* Every power series is an immediate continuation of itself since $a_n=\dfrac{P_a^{(n)}(a)}{n!}$ where $Pa(z)$ is the power series (25)

(ii) there exists a finite sequence $0=t_0 < t_1 < t_2 \ldots < t_n < t_{n+1}=b$ such that the part C_v $(v=0, 1, 2, \ldots, n)$ of the curve (30) corresponding to $t_v \leqslant t \leqslant t_{v+1}$ is a rectifiable simple Jordan curve, closed or not, $u(t)$ and $v(t)$ being themselves *continuous* functions of t in $0 \leqslant t \leqslant 1$. The whole curve (30) is therefore rectifiable. When $a=b$, we say L (a, b) is a *closed* path. When (30) itself is a simple rectifiable Jordan curve we say that the path is *simple* and when, in addition it is closed, we say it is a *simple closed* path. We attach a definite direction called the positive direction from a to b as follows. We start from $z=a$ and describe C_0 in the direction of increasing arc measured from $z=a$; then we start from $u(t_1)+iv(t_1)$ and describe C_1 in the direction of increasing arc measured from this point and so on until we arrive at the point $z=b$ on the simple Jordan curve C_n.* Thus we can make a point z describe the path L (a, b) from $z=a$ to $z=b$ in the positive direction in such a manner that the length $l\,(a, z)$ of the curve (30) from $z=a$ to $z=z$ *increases continuously* from $l\,(a, a)=0$ to the total length $l\,(a, b)=L$ of the path.

(7·5.) Suppose now that L (a, b) is a path joining the two points $z=a$ and $z=$b. We say that a point z_0 of L (a, b) is *an accessible point with respect to the element* $P_a\,(z)$ *of an analytic function* $f\,(z)$ if there exists a finite number $(P_a=P_{a_0},\ P_{a_1}\ P_{a_2} \ldots,\ P_{a_n})$ of elements of $f\,(z)$ satisfying the following conditions.

(i) P_{a_v} is an immediate continuation of $P_{a_{v-1}}$, $v=1, 2, \ldots, n$;

(ii) every point of the arc from a to z_0 (inclusive) of the path L (a, b) is an interior point of at least one C (P_{a_v}).

(7·6) From the above definition of an accessible point it is possible to deduce, by using well-known properties of closed point sets and the definition of immediate continuation, that the points $a_1, a_2, \ldots, a_n$ can be chosen so as to lie on L (a, b) itself, that is, the centres of the elements can be chosen on L (a, b). When this is done we say that the continuation of $P_a\,(z)$ from a to z_0 is effected *along* the path L (a, b). It may be noted that at every accessible point z_0 there is an element of $f(z)$ obtained from P_{a_n} by the rule (26).

(7·7) Suppose, now, that we are given an element $P_a\,(z)$ of $f\,(z)$, a point b and a path L (a, b). It may happen that all points of the arc L (a, b) from a to b are accessible points. If not, there is at least one point on L (a, b) which is *not* accessible with respect to the

* When one of the curves c_v is closed we have the choice of two distinct direction to describe c_v starting from $u\,(t_v)+iv\,(t_v)$. In this case we agree to choose the positive direction along this contour as explained in § 3·4.

element $P_a(z)$. Let z be an accessible point and let $l(a, z)$ be the length of the arc of $L(a, b)$ from a to z measured from $z=a$ as explained in (7·4.) Then all points of $L(a, b)$ between a and z are also accessible. Consider the upper bound of the numbers $l(a, z)$ where z is an accessible point. If l be this upper bound, there is a point z_0 on $L(a, b)$ such that $l, (a, z_0)=l$ since $l(a, z)$ is a continuous function of z on $L(a, b)$. It follows that all points of $L(a, b)$ from a to z_0, z_0 excluded, are accessible with respect to $P_a(z)$ and that z_0 is *not* an accessible point. We shall say that z_0 is a *singular point* of $f(z)$. *There cannot exist an element* $P_{z_0}(z)$ *of* $f(z)$ *which is a continuation of* $P_a(z)$ *along the path* $L(a, b)$. Conversely, if all points except z_0 of the arc (a, z_0) of $L(a, b)$ are known to be accessible while there exists no element $P_{z_0}(z)$ which is a continuation of $P_a(z)$ along $L(a, b)$ then z_0 is a singular point of $f(z)$. It is important to note that a singular point is specified only when the *starting element* $P_a(z)$ and the path *along* which $P_a(z)$ is continued are specified. We shall show later, by examples, that a point z_0 may be a singular point for a certain starting element and for a certain path but may cease to be so if either the starting element or the path is altered.

8. *Uniform and multi-form or multiple-valued functions.*

Suppose that all points of a closed path $L(a, a)$ (including 'a' regarded as the second end-point) are accessible points for an element $P_a(z)$ of a function $f(z)$. It can happen that the element $Q_a(z)$ which we shall obtain by continuing $P_a(z)$ along $L(a, a)$ in the positive direction back to a is quite distinct from $P_a(z)$. In this case we say that $f(z)$ is multiform or multiple-valued. If we continue $Q_a(z)$ along $L(a, a)$ in the negative direction we shall arrive back at $z=a$ with $P_a(z)$ but if $Q_a(z)$ be continued along $L(a, a)$ again in the positive direction we *might* obtain an element distinct from both P_a and Q_a. Now, consider *all* closed paths at $z=a$ along which $P_a(z)$ can be *continued back to* a. Suppose in all these cases we come back to a with the self-same element $P_a(z)$ and that this is true for *all* elements $P_a(z)$ of $f(z)$. Then we say that $f(z)$ is a *uniform analytic function.*

(8·1) Before proceeding further, we shall illustrate the notions introduced above by means of simple examples. The function $f(z)=\dfrac{1}{1-z}$ has one and only one element at every point of the plane except $z=1$ and therefore is a uniform analytic function in the whole complex plane (§ 2·3) except $z=1$ which is a singular point according to the definition of (§ 7·7.) Next consider $f(z)=\sqrt{z}$. Let

$z = re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. The function $\sqrt{z}$ takes two values at every point except $z=0$ and ∞ and has two distinct elements at every such point. Consider the point $z=r$. Let E_1* and E_2 be the two elements corresponding to $f(r) = +\sqrt{r}$ and $-\sqrt{r}$ at $z=r$. If we start with E_1 at $z=r$ and continue along a simple closed path whose interior does not contain $z=0$ we shall arrive back at $z=r$ with E_1; but if the interior contains $z=0$ we shall come back to E_2. The function $\sqrt{z}$ is a two-valued analytic function having two elements, negatives of each other, at every point of the plane except $z=0$ and ∞ which are its two singular points. We now pass to a more complicated example given by

$$f(z) = \sqrt{(\sqrt{z^2 - 2z} + i)} \quad \ldots \quad (31)$$

which take four values at every point except $z=0$, 1, 2 and ∞ where one of the radicals vanishes or becomes infinite. At a point $z=r>2$ there are four distinct elements E_1, E_2, E_3 and E_4 corresponding to the four values $+\sqrt{(\sqrt{r^2-2r}+i)}$, $-\sqrt{(\sqrt{r^2-2r}+i)}$, $+\sqrt{(-\sqrt{r^2-2r}+i)}$ and $-\sqrt{(-\sqrt{r^2-2r}+i)}$ of $f(z)$ at $z=r$. If we start with E_1 and continue along a simple closed path $C(r, r) = C_r$ whose interior $I(C_r)$ contains none of the points $z=0$, 1 or 2, we shall arrive back to the element E_1; if $I(C_r)$ contains $z=2$ but not $z=0$ or 1 we shall obtain the element E_3 when we return to $z=r$; if $I(C_r)$ contains $z=2$ and 1 but not $z=0$ we shall come back to E_4; and if $I(C_r)$ contains all the three points $z=0$, 1 and 2 we shall come back to E_2. The points $z=0$ and 2 are singular points with respect to any one of the starting elements at $z=r$ and for any path $L(r, 0)$ or $L(r, 2)$ all of whose points except $z=0$ or 2, as the case may be, are accessible points. The point $z=1$ is a singular point with respect to the element E_3 or E_4 along any simple path $L(r, 1)$; but $z=1$ is *not* a singular point with respect to E_1 or E_2 along such a path. But $z=1$ *is* a singular point with respect to E_1 or E_2 along a path $L(r, 1)$ composed of a simple Jordan arc from $z=r$ to $z=\frac{1}{2}(r+2)$ followed by a contour starting and ending at $z=\frac{1}{2}(r+2)$ whose interior contains $z=2$ (but not $z=0$ or 1) which again is followed by a simple Jordan arc from $z=\frac{1}{2}(r+2)$ to $z=1$, all these curves being supposed rectifiable. Here we see an illustration of the remarks at the end of § 7·7.

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* E_1 is the power series $P_r(z) = +\sqrt{r}\left\{1 + \frac{z-r}{2r} - \frac{(z-r)^2}{8r^2} + \cdots\right\}$ which is the binomial expansion of $\sqrt{r}\left\{1 + \frac{z-r}{r}\right\}^{\frac{1}{2}}$; E_2 is the negative of the above series. Both are valid in $|z-r| < r$.

(8.2.) *Domain of existence of analytic functions according to Weierstrass.* We have seen the importance of the domain of definition of an analytic function according to Cauchy. In the view point adopted by Weierstrass, the domain of definition of a function is implicit in the definition of the function itself. We consider the totality of the elements which constitute the function. Each element has a circle of convergence in the interior of which the value of the function is defined by that of the power series representing the element. *The set of interior points* of the *totality* of the circles of convergence of *the various elements constituting a function* is called the *domain of existence* * of the function. If the function be uniform, there is *one* and *only one* element with *centre* at a given point of this domain of existence and therefore, by the very nature of the process of continuation, the domain of existence is *an open connected set*, that is, *a domain* (§ 1·4). Even if the function be multi-form, the set of interior points of the various circles of convergence of the elements of the function constitute *a domain considered as a point set in the complex plane.* But at a given point of this domain there will be as many circles of convergence as there are distinct elements with that point as centre and therefore we shall have to regard *such portions of the plane as covered more than once in order to get a complete idea of the domain of existence of multi-form functions.* This fact is the basis of the theory of Riemann Surfaces but we do not propose to enter into a detailed discussion of the theory at present.

(8.3.) *Classification of singularities.* We have already emphasised the fact that a singular point can be completely specified only in relation to the starting element and the path followed in effecting the continuation of that element. But in the case of uniform functions no such complications arise. A singular point with respect to one starting element and a certain path *remains so* if we alter the element or the path. Hence, in the case of uniform functions, the notion of singular points is free from the *individual* elements or the path followed in the process of continuation of these elements. *A point which is the boundary point of the domain of existence of an uniform analytic function is a singular point of the function and such points are its only singular points.* A singular point z_0 of an uniform function is said to be isolated if there exists a $C(z_0, < r)$ which contains no singular point of the function other than z_0. An isolated singularity z_0 is said to be a pole if $|f(z)| \to \infty$ as $z \to z_0$. All singular points of a uniform function other than poles are said to be an essential singularities.

* We reserve the term 'domain of definition' for functions analytic according to Cauchy. We shall indicate the relation between the two later on (see § 9.)

Any point of the domain of existence of a uniform function is said to be a *regular or ordinary point*. An isolated singularity z_0 is such that there exists a $C(z_0, < r)$ all points of which, except $z = z_0$, are regular points. If $z = a$ be a point of the domain of existence of a uniform analytic function $f(z)$, then the radius of convergence of the element $P_a(z)$ of $f(z)$ at $z = a$ given by

$$P_a(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \qquad \ldots \quad (32)$$

is *equal* to the *minimum distance* of the point $z = a$ from the points of the set $S(f)$ of the singularities of $f(z)$. The only uniform function with no singularities in the whole complex plane (§ 2·3) is a constant and conversely. A function with only *a finite number of poles in the whole complex plane* is *a rational function and conversely*. If the only pole of a uniform function be $z = \infty$ it is a polynomial and conversely. If a uniform function has its pole singularity at $z = \infty$, the function is said to be *an integral function* and if $z = \infty$ be *not* a pole but is an isolated essential singularity the function is called *a transcendental integral function*. The exponential function e^z, the trigonometric functions $\sin z$ and $\cos z$ are examples of transcendental integral functions. In the case of integral functions, the series (32) converges in the whole finite plane (§ 2·3) whatever the point a may be.*

8·4. In the case of multiform functions we can formulate definitions similar to those of § 8·3 paying due regard to the starting element and the path followed. An exhaustive discussion would lead us too far but we note, in passing, a particular type of singularity peculiar to multiform function. Let z_o be an isolated singularity of a multiform functions $f(z)$; z_o is a singular point with respect to at least one starting element and a corresponding path and there are no other singular points other than z_o in some $C(z_o, r)$. The singular point z_o is said to be a *critical or branch point of* $f(z)$ if there exists a $C(z_o, < r)$ which contains a point $z = a$ at which there is more than one element and every element of $f(z)$ with centre in $C(z_o, < r)$ can be obtained from *any one element* by the process of continuation effected *along suitable paths lying in* $C(z_o, < r)$. For instance, $z = 0$ and ∞ are critical points of the two-valued function $f(z) = \sqrt{z}$; the points $z = a$ and ∞ are critical points for the infinite-valued function $\log(z-a)$; and the points $a_1, a_2, \ldots, a_v$ are critical points for

$$\sqrt{(z-a_1)(z-a_2)\ldots(z-a_v)}.$$

* We shall presently discuss the relation between poles, etc., defined in § 8·3 with those defined in § 5·1 – § 5·6.

8·5. There are two general results on singularities which we wish to mention. If $P_a(z)$ be an element of a function $f(z)$ *there is at least one singularity of* $f(z)$ *on the circumference* $C(P_a)$ *of the circle of convergence* $C(P_a)$ *of* $f(z)$. The second result concerns only multiple-valued functions. Let $L_1(a, b)$ and $L_2(a, b)$ be two simple paths joining a and b, the two paths together forming a contour C whose interior we denote by $I(C)$. Let $P_a(z)$ be an element of a function $f(z)$ at $z=a$ which when continued along L_1 and L_2 lead to distinct elements at $z=b$. Then *there is at least one singular point of* $f(z)$ *in* $I(C)$.

9. *Relation between Cauchy's and Weierstrass's definitions of analytic functions.* Firstly we can state that *a uniform analytic function according to Weierstrass is a single-valued analytic function according to Cauchy, the domain of existence being its domain of definition* (§ 2·2); this is because a power series is a single-valued analytic function according to Cauchy in its circle of convergence (§ 4). When we consider multiform functions we cannot make a similar statement without developing the theory of multiform functions based on Cauchy's definition but such a theory will have to make use of some idea of continuation similar to that which is the basis of Weierstrass's method of approach to analytic functions. On the other hand, the properties of single-valued analytic functions according to Cauchy discussed in §2 – §6 are so fundamental that it will be a great loss to the theory of analytic functions if they could not be adopted for applications to the case of multiform or many-valued analytic functions. For this purpose we introduce the notion of *incomplete Weierstrassian analytic functions.* Let $P_a(z)$ be an element of $f(z)$ and let D be a domain containing $z=a$ in its interior. Consider the totality of the continuations of $P_a(z)$ which are obtainable by continuing $P_a(z)$ *along paths lying in D.* We shall call this totality *an incomplete analytic function with respect to D of the given function* $f(z)$ and we shall denote it by $f(z, D, P_a)$. Given D, there can be more than one incomplete function if there be more than one element of $f(z)$ at any point of D. But given D and $P_a(z)$, the incomplete function $f(z, D, P_a)$ is quite determinate. When $f(z)$ is uniform, all its incomplete functions are single-valued in D whatever D or P_a be. But, even if $f(z)$ is multiform, we can, in many cases, *so choose D that* $f(z, D, P_a)$ *is single-valued in D*; we can also say that $f(z, D, P_a)$ *is uniform with respect to D.* If there are several single-valued (or uniform) incomplete functions of $f(z)$ with respect to D we shall call each of these a branch of $f(z)$ in D. When $f(z, D, P_a)$ is uniform with respect to D, we can start with any

element of $f(z, D, P_a)$ in D and if we continue this element back to its centre by a *path lying in D* we shall always come back to the element with which we started. For instance, consider $f(z) = \sqrt{z}$. At $z = r > 0$ there are two distinct elements E_1 and E_2 corresponding to the two values $f(r) = \sqrt{r}$ and $-\sqrt{r}$. The incomplete functions $f(z, D, E_1)$ and $f(z, D, E_2)$ are both single-valued or uniform in D when D is any simply connected domain not containig $z = 0$.

(9.1.) Now, let $f(z, D, P_a)$ be a uniform incomplete function of a function $f(z)$ with respect to D. Let D (f) be that portion of D every point which is an interior point of the circle of convergence of an element of $f(z, D, P_a)$. Then we can say *that $f(z, D, P_a)$ is a single-valued analytic function according to Cauchy in the domain D (f)*. In the example $f(z) = \sqrt{z}$ discussed in § 9, D (f) coincides with D so that $f(z, D, E_1)$ and $f(z, D, E_2)$ are both single-valued analytic functions according to Cauchy in the domain D; here is an example when there are two incomplete uniform functions in given domain D, that is, there are two branches of $f(z)$ in D.

(9.2.) We have seen that a uniform analytic function $f(z)$ according to Weierstrass is single-valued and analytic according to Cauchy in the domain D which is the domain of existence of $f(z)$. In this case a pole or an isolated essential singularity according to § 7·7 and § 8·3 is also a pole or an isolated essential singularity according to the notions introduced in § 5·1 and § 5·2. Also any point of D is a regular point in the sense of § 5. Similar statements hold for any uniform incomplete function of a given multiform function.

(9.3.) Consider, now, a function $f(z)$ single-valued and analytic in the domain D, in the sense of Cauchy. At each point of D there is an element of $f(z)$ [§ 4·1]. Let $P_a(z)$ be an element of $f(z)$ at $z = a$ in D. By using equation (13) we can show that any element of $f(z)$ in D is a continuation of $P_a(z)$ and that if we start with any element and continue back to the centre of the element by any path lying in D we shall come back to the original element. It follows that $f(z) = f_1(z, D, P_a)$ is an uniform incomplete function of the analytic function $f_1(z)$. according to Weierstrass, determined by the element $P_a(z)$ [or any other element of $f(z)$ with centre in D]. Moreover $D = D(f_1)$ (§ 9·1). *Thus a single-valued analytic function according to Cauchy defined in a domain is always a uniform incomplete Weierstrassian function. A single-valued analytic function according to Cauchy is an analytic function in the sense of Weierstrass* (§ *7·3*) *if and*

only if the Weierstrassian function arising from any element of the given function is a uniform analytic function (§ 8) *and its domain of existence coincides with the domain of definition of the given function.* But *in all cases* a single-valued analytic function according to Cauchy with *a domain as its domain of definition* is a uniform incomplete Weierstrassian analytic function.

(9.4.) But the matter is not so simple when the domain of definition S of a single-valued analytic function $f(z)$ is *not a domain.* When S consists of more than one domain we cannot make any statement such as those in § 9·2 and § 9·3. For instance, let D_1 and D_2 be two domains such that $D_1 + B(D_1)$ and $D_2 + B(D_2)$ have no common points. Let.

$$f(z) = \begin{cases} z & \text{in } D_1 \\ z^2 & \text{in } D_2 \end{cases} \qquad \ldots \quad (33)$$

Then $f(z)$ is single-valued and analytic on $S = D_1 + D_2$. But no element of $f(z)$ in D_2 can be a continuation of any element of $f(z)$ in D_1. The Weierstrassian function, of which $f_1(z) = z$ in D_1 is an incomplete function, is the polynomial $P_1(z) = z$ and which is uniform in the whole finite plane; similarly for $f_2(z) = z^2$ in D_2, the corresponding Weierstrassian function is $P_2(z) = z^2$ in the whole finite plane. And $P_1(z)$ and $P_2(z)$ cannot be connected by any process of continuation of the elements of either of these functions. Still there is an essential unity in such functions as (33) since one and the same sequence $\{P_n(z)\}$ of polynomials can be found such that

$$\lim_{n \to \infty} P_n(z) = \begin{cases} z & \text{in } D_1 \\ z^2 & \text{in } D_2 \end{cases} \qquad \ldots \quad (34)$$

(9.5.) If we consider the relative advantages and disadvantages of the two definitions of analytic functions, we can say that Cauchy's definition does not give a complete idea either of the function or its singularities when the domain of definition is a single domain; while Weierstrass's view point gives such an idea. But when we have to consider the simultaneous properties of distinct analytic functions defined in domains without common points Cauchy's definition enables us to consider the whole problem as that of a single analytic function [such, for instance, is the relation represented by (34);] in this case Weierstrass' view point does not apply at all. Again multiform functions are not easily treated by Cauchy's method while it is very natural for multiform functions to present themselves from Weierstrass' view point. For practical purposes a judicial combination of the two points of view is the most useful and has contributed to the rapid and extensive developments of the theory of analytic functions.

10. The class of analytic functions in a given set S.

Let us consider the class of all single-valued analytic functions defined on a set S. We shall denote the class by $C_A(S)$. If $f_1(z)$ and $f_2(z)$ belong to $C_A(S)$, then $c_1 f_1(z) + c_2 f_2(z)$, c_1 and c_2 being constants, and $f_1(z)$, $f_2(z)$ belong to $C_A(S)$. When S is an open set $f_1(z) \div f_2(z)$ belongs to $C_A(S)$ provided $f_2(z) \neq 0$ on S. Lastly, if $\{f_n(z)\}$ is a sequence of functions of $C_A(S)$ converging uniformly to a function $f(z)$ on *any* closed set in S, $f(z)$ belongs to $C_A(S)$ provided S is an open set. The last two properties might not hold, when S is an arbitrary set.

(10.1) *Uniqueness of analytic functions.* Let $f_1(z)$ and $f_2(z)$ be two single-valued analytic functions in a domain D. Let L be a sequence $\{z_n\}$ of points such z_n belongs to D and L has at least one limit point interior to D. Then if $f_1(z_n) = f_2(z_n)$, $n = 1, 2, \ldots$, then $f_1(z) \equiv f_2(z)$ for all z in D. If $z = a$ be a point interior to D and $f_1(a) = f_2(a)$, $f_1'(a) = f_2(a), \ldots, f_1^{(n)}(a) = f_2^{(n)}(a) \ldots$ then $f_1(z) \equiv f_2(z)$ for all z in D. The first property follows from the fact that the points where $f(z) = 0$, $f(z)$ being single-valued and analytic in a domain D, form an isolated set (§ 1·2). The second follows from the equation (4) and the results of § 9·3. From the second result we deduce that the two power series (25) and (26) (§ 7·1) are equal *not only* in $|z-b| < \rho - |b|$ but also *at all points common* to the circles of convergence of (25) and (26).

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GLEANINGS

THE WORLD *Sans* MATHEMATICS

Let us imagine, if we can, that by some mighty cataclysm there should be wiped off of the face of the Earth to-night every book on mathematics, every mathematical symbol of any kind, every written page or printed sheet upon which a trace of mathematics appears ...... Every mill in the whole world would slow down and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measures, its tables, and its computing machinery. Every ship in the seven seas would be stricken with blindness and would wallow helpless, awaiting the probable starvation of its human burden. Not a rivet would be driven in a skyscraper in New York City, because the steel girders would have lost their numbers. Wall street would close its portals; the engineering world would waken to-morrow morning to a living death; the mines would shut down and trade would relapse to the condition of barter as in the days of savagery. It is a picture so ridiculous that we smile at the very impossibility. But it is a real picture, ridiculous though it seems, – a picture of the world sending forth the SOS for help, a hurry call for the return of mathematics that many take so much pleasure in condemning.

D. E. SMITH as quoted in *The Mathematics Teacher*, Vol. 29 No. 6.

SOME MORE FORMULAE FOR THE BERNOULLIAN NUMBERS

BY

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1. In a previous paper published in the *Mathematics Student* Vol. III (141-151), the author obtained several recurrence formulae for the Bernoulli's numbers with the help of the theory of symmetric functions and the determinants. In this note a few more formulae are obtained.

2. Let

$$f(x)=(1-ax)(1-bx)(1-cx)\cdots\cdots(1-wx)$$
$$=a_0-a_1x+a_2x^2-a_3x^3+\cdots\cdots+(-1)^r a_r x^r+\cdots \qquad \ldots (1)$$

then

$$[f(x)]^{-1}=h_0+h_1x+h_2x^2+\cdots\cdots+h_rx+\cdots \qquad \ldots (2)$$

where a_r and h_r, respectively, denote the elementary and the complete homogeneous (or aleph) symmetric functions of the variables a, b, c, $\cdots w$, of degree r. We write $S_r=\sum a^r$.

From (1) and (2), Wronski's relation connecting a_r and h_r follows, i.e.,

$$a_0\,h_r-a_1\,h_{r-1}+a_2\,h_{r-2}-a_3\,h_{r-3}+\cdots\cdots+(-1)^r\,a_r\,h_0=0.$$

Determinantally we can write

$$a_n=\begin{vmatrix} h_1 & h_2 & \cdots\cdots\cdots & h_n \\ h_0 & h_1 & \cdots\cdots\cdots & h_{n-1} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ 0 & 0 & \cdots\cdots\cdots & h_1 \end{vmatrix} \qquad \ldots (A)$$

From Newton's well-known formula

$$S_r-a_1\,S_{r-1}+a_2\,S_{r-2}-a_3\,S_{r-3}+\cdots\cdots+(-1)^r r a_r=0,$$

S_n may be expressed in the form

$$S_n=\begin{vmatrix} a_1 & 1 & 0 & 0 \cdots & 0 \\ 2\,a_2 & a_1 & 1 & 0 \cdots & 0 \\ 3\,a_3 & a_2 & a_1 & 1 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n\,a_n & a_{n-1} & a_{n-2} & a_{n-3},\cdots & a_1 \end{vmatrix} \qquad \ldots (B)$$

Also when n is odd *

$$S_n = \begin{vmatrix} h_1 & 1 & 0 & \cdots\cdots & 0 \\ 2\,h_2 & h_1 & 1 & \cdots\cdots & 0 \\ 3\,h_3 & h_2 & h_1 & \cdots\cdots & 0 \\ \ldots & \ldots & \ldots & \cdots\cdots & \\ n\,h_n & h_{n-1} & h_{n-2}, & \cdots\cdots & h_1 \end{vmatrix} \quad \ldots \text{(C)}$$

When n is *even*, the sign of the determinant is negative

3. It is known that †

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2\,!} - B_2 \frac{x^4}{4\,!} + B_3 \frac{x^6}{6\,!} - B_4 \frac{x^8}{8\,!} + \cdots\cdots; \quad \ldots (3)$$

where B's are the Bernoullian numbers so that

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \text{ etc.}$$

Comparison of (1) and (3) expresses the a's in terms of the B's, so that

$$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{B_1}{2\,!},\ a_3 = 0,\ a_4 = \frac{-B_2}{4\,!},\ \cdots\cdots$$

$$a_{2n} = (-1)^{n+1}\,\frac{B_n}{(2n)\,!}; \quad a_{2n+1} = 0.$$

Since $\dfrac{e^x - 1}{x} = 1 + \dfrac{x}{2\,!} + \dfrac{x^2}{3\,!} + \cdots\cdots + \dfrac{x^r}{(r+1)\,!} + \cdots\cdots; \quad \ldots (4)$

therefore comparing (2) and (4) we have $h_r = \dfrac{1}{(r+1)\,!}$.

4. The recurrence relations for the Bernoullian numbers will now be considered.

From (A) we can express $B_{n/2}$ as a Recurrent, ‡ i.e.

$$(-1)^{n/2+1} B_{n/2} \div n\,! = \begin{vmatrix} \frac{1}{2\,!} & \frac{1}{3\,!} & \frac{1}{4\,!} & \cdots\cdots & \frac{1}{(n+1)\,!} \\ 1 & \frac{1}{2\,!} & \frac{1}{3\,!} & \cdots\cdots & \frac{1}{n\,!} \\ \ldots & \ldots & \ldots & \cdots\cdots & \ldots \\ \ldots & \ldots & \ldots & \cdots\cdots & \ldots \\ 0 & 0 & 0 & \cdots\cdots & \frac{1}{2\,!} \end{vmatrix}$$

* See Muir, *History of Determinants* Vol. III, P. 246.

† Whittaker and Watson *Modern Analysis* (1920) P. 127.

‡ Compare also Milne-Thompson '*The Calculus of Finite Differences*' (1933) P. 139, where the result is a bit differently stated.

From (C), when n *is even*,

$$S_n = -\begin{vmatrix} \frac{1}{2\,!} & 1 & 0 & \cdots\cdots & 0 \\ \frac{2}{3\,!} & \frac{1}{2\,!} & 1 & \cdots\cdots & 0 \\ \frac{3}{4\,!} & \frac{1}{3\,!} & \frac{1}{2\,!} & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \frac{n}{(n+1)\,!}, & \frac{1}{n\,!}, & \frac{1}{(n-1)\,!}, & \cdots\cdots & \frac{1}{2\,!} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\,!} & 1 & \cdots\cdots & 0 \\ \frac{1}{3\,!} & \frac{1}{2\,!} & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \frac{1}{(n+1)\,!}, & \frac{1}{n\,!} & \cdots\cdots & \frac{1}{2\,!} \end{vmatrix}$$

on subtracting the second column from the first.

Therefore $S_n = (-1)^{n/2+1} B_{n/2} \div n\,!$, S_1 being $\frac{1}{2}$.

When n is odd and exceeds unity, $S_n = 0$.

From (B) following the method given in my paper * referred to above, and considering the value of S_n, we arrive at the recurrence relation :—

$$\frac{1}{2} + \sum_{t=1}^{n/2} (-1)^t \binom{2t^n - 1} B_t = (-1)^{n/2+1} B_{n/2} \,;$$

where n is even ; and when n is odd the right hand side is zero.

* Zia-ud-Din, *Math. Student* Vol. III (1936), p. 145. Art. 6.

ON THE NATURE OF THE CONICS OF A FOUR-LINE SYSTEM WHICH PASS THROUGH A GIVEN POINT.

BY

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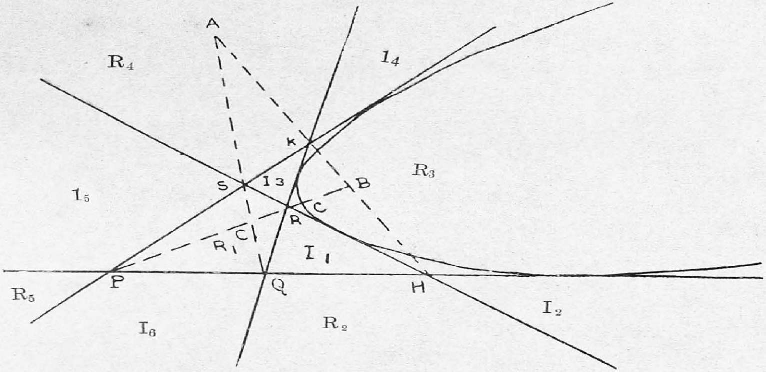
1. A singly infinite number of conics can be drawn to touch four given straight lines, two members of which will pass through any point in the plane. The object of the paper is to examine the nature of these two conics. The four lines divide the plane into eleven regions; the nature of the conics through a point depends on the position of the point in one of the regions. The situation of the two rectangular hyperbolas of the system is also discussed.

2. Let the four given real lines form a non-re-entrant quadrilateral PQRS and let ABC be the diagonal triangle of the above quadrilateral, the vertex C being inside. Let ABC be taken as the triangle of reference, the system of co-ordinates being areal. It is easily seen that the equations of PQ, QR, RS and SP are respectively



$$L_1 \equiv lx + my + nz = 0$$
$$L_2 \equiv lx - my + nz = 0$$
$$L_3 \equiv lx + my - nz = 0$$
$$L_4 \equiv -lx + my + nz = 0$$

l, m, n being of the same sign, say positive. Also as l and n are proportional to the perpendiculars from A, C on PQ we have for our figure $l > n$. Similarly $l > m$ and $m > n$. Hence we have $l > m > n$.

Let the equation of the system of conics be $ax^2+by^2+cz^2=0$. Where $l^2/a+m^2/b+n^2/c=0$. Eliminating a and writing $\lambda=b/c$ the system can be represented by the equation

$$\lambda^2n^2y^2+\lambda(n^2z^2+m^2y^2-l^2x^2)+m^2z^2=0 \qquad \ldots \quad (2\cdot1)$$

This equation gives two values of λ for a given point T (x, y, z) and the two values are of the same sign if x, y, z are real. The two conics through T are real, co-incident or imaginary according as $L_1 L_2 L_3 L_4 <$ = or >0. ... (2·2)

It thus follows that if T lies on one of the four given lines, the conics coincide and if it lies in one of the regions marked R_1, R_2, R_3, R_4, R_5 the conics are real; whereas if it lies in one of the regions marked I_1, I_2, I_3, I_4, I_5, I_6 the conics are imaginary.

3. The conic (2·1) will for a particular λ be an ellipse or a hyperbola according as the roots of the quadratic in y/z namely

$$\lambda^2n^2y^2+\lambda\{n^2z^2+m^2y^2-l^2(y+z)^2\}+m^2z^2=0$$

are imaginary or real i.e., according as

$$\lambda(\lambda n^2+m^2)\left\{\lambda(n^2-l^2)+(m^2-l^2)\right\} \gtrless 0. \qquad \ldots \quad (3\cdot1)$$

We have noted above that the two values of λ for a real point x, y, z are of the same sign. This sign is negative when

$$n^2z^2+m^2y^2-l^2x^2\equiv\tfrac{1}{2}(L_1L_4-L_2L_3)$$

is positive and *vice versa*. In R_1 we have L_1, L_2, $-L_3$, L_4 are all positive; in R_3, L_1, $L_4>0$ and $L_2, -L_3<0$. in R_5 L_1, $L_4<0$ and $L_2, -L_3>0$. Hence both values of λ are negative in R_1, R_3, R_5. It can be similarly seen that both values of λ are positive in R_2, R_4. It therefore follows that for a point in R_2, R_4 (where $\lambda>0$) the two conics are hyperbolas since we have supposed $l>m>n$.

We have yet to examine the regions R_1, R_3, R_5 for which λ is negative. Writing $\lambda n^2+m^2=\rho$ the equation (2·1) transforms into

$$y^2\rho^2+\rho(L_2L_4-L_1L_3)/2+m^2l^2x^2=0$$

Remembering the signs of L_1, L_2, L_3, L_4 in R_1, R_3, R_5 it can be easily seen that $\rho<0$ in R_1, and that $\rho>0$ in R_3, R_5. Thus for a point in R_1 the conic is an ellipse or a hyperbola according as

$$\lambda(n^2-l^2)+(m^2-l^2) > \text{ or } < 0$$

i.e. according as $\lambda \lessgtr \dfrac{m^2-l^2}{l^2-n^2}=\lambda_0$ (say). For a point in R_3, R_5 the inequality is to be reversed.

4. Now if λ_1, λ_2 are the roots of (2·1)

$$(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0) \equiv \lambda_1 \lambda_2 - \lambda_0(\lambda_1 + \lambda_2) + \lambda_0^2$$
$$\equiv \frac{l^2}{n^2 y^2 (l^2 - n^2)^2} \sum x^2 (l^2 - m^2)(l^2 - n^2).$$

The conic $\sum x^2(l^2 - m^2)(l^2 - n^2) = 0$ is the unique parabola $\sum$ of the system and corresponds to the value λ_0 of λ. Hence for a point on the positive side of this parabola, the conics are both ellipses or hyperbolas. Within the parabola, however one conic is an ellipse, the other a hyperbola. Upon the parabola one of the two conics is the parabola itself, the other may be an ellipse or a hyperbola. Where the parabola touches the four lines, the conics coincide with the parabola itself. We have, thus, the theorem

"*Within the parabola touching the four lines, the two conics through a point are one an ellipse and the other a hyperbola.*"

It is evident that for our figure, the parabola lies in the region R_3 —— B lies within, A and C outside.

5. Outside the parabola λ_1, λ_2 are both greater or both less than λ_0 (accordingly the two conics are both hyperbolas or ellipses) according as

$$\lambda_1 + \lambda_2 - 2\lambda_0 \equiv -\left\{ \frac{n^2 z^2 + m^2 y^2 - l^2 x^2}{n^2 y^2} + 2\,\frac{m^2 - l^2}{l^2 - n^2} \right\}$$
$$\equiv -\frac{1}{n^2 y^2 (l^2 - n^2)} \left\{ -l^2(l^2 - n^2)x^2 + y^2(l^2 m^2 + m^2 n^2 - 2l^2 n^2) + n^2(l^2 - n^2)z^2 \right\}$$

is positive or negative. This brings in the new conic :—

$$C \equiv -l^2(l^2 - n^2)x^2 + (l^2 m^2 + m^2 n^2 - 2l^2 n^2)y^2 + n^2(l^2 - n^2)z^2 = 0.$$

It is a hyperbola having ABC for a self-conjugate triangle, and passing through Q, S and the points of contact of the Parabola with the given lines. And as

$$2m^2 C \equiv l^2(m^2 - n^2)(L_1 L_4 - L_2 L_3) + n^2(l^2 - m^2)(L_2 L_4 - L_1 L_3)$$

C preserves its sign which is positive in R_1 and negative in R_2, R_4. Hence in R_1, $\lambda < \lambda_0$ and the two conics are both ellipses for a point in R_1. As the conic C meets PQ, PS at Q, S and the points of contact of the parabola, R_5 is on the same side of C as R_1. Thus in R_5, $\lambda < \lambda_0$ and the two conics through a point are hyperbolas.

6. We have seen for a point in R_3, within the parabola one conic is an ellipse, the other a hyperbola. Outside the parabola and

within R_3 there are five sub-regions. Remembering that the hyperbola C passes through the points of contact of the parabola, it is evident that two of these sub-regions—at H, K—lie on the negative side of C, the other three on the positive side. In these two sub-regions $\lambda > \lambda_0$ and therefore both the conics are ellipses.

Similarly in the other three sub-regions the conics are both hyperbolas. Again if the side CB of the triangle ABC intersects the parabola $\sum$ in c, b, and if we take a point X in cb, the line cb must be regarded as the thin hyperbola through X and hence all the ellipses in R_3 cuts $x=0$ within the parabola and the hyperbolas outside. Similarly all the hyperbolas intersect AB within and the ellipses outside the parabola. It is also evident that any two ellipses in R_3 intersect in the two sub-regions at H, K and the hyperbolas in the other three sub-regions.

7. Summing up we have

(1) *For a point in R_1, both the conics are ellipses ;*

(2) *For a point in R_2, R_4, R_5 the conics are both hyperbolas* ;

(3) *For a point in R_3 within the parabola $\sum$ one conic is an ellipse, the other a hyperbola. Outside the parabola, in two of the sub-regions namely those near H and K the conics are both ellipses, whereas in the other three they are both hyperbolas.*

8. We shall next discuss the position of the rectangular hyperbolas of the system. It can be shown that if α, β, γ are the sides of the triangle ABC, the conic given by (2·1) is a rectangular hyperbola if $\lambda^2\beta^2n^2 + \lambda\{\beta^2m^2 + \gamma^2n^2 - \alpha^2l^2\} + \gamma^2m^2 = 0$ from which it immediately follows that "The two rectangular hyperbolas belonging to a four line system of conics intersect on the in-centre and the ex-centres of the diagonal triangle."

The rectangular hyperbolas will be real coincident or imaginary according as

$$\sum \alpha^4 l^4 - 2\sum \beta^2\gamma^2 m^2 n^2 > = \text{ or } < 0.$$

(i.e.) according as $-P_1P_2P_3P_4 > =$ or < 0 where P_1, P_2, P_3, P_4 are the perpendiculars from the in-centre I on L_1, L_2, L_3, L_4. Hence the in-centre and therefore the ex-centres must lie in one of the regions R_1, R_2, R_3, R_4, R_5 if the rectangular hyperbolas are real. If the in-centre lies on one of the given lines, the ex-centres will lie on the other three lines, and then the rectangular hyperbolas coincide.

Again it is evident from the figure that I must lie in R_1 or R_2 or R_4. If it lies in R_1, by considering the signs of the expressions

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$\pm l\alpha \pm m\beta \pm n\gamma$ it can be seen that I_A, I_B lie in R_1 but I_C lies in a region which makes all the expressions L_1, L_2, $-L_3$, L_4 negative. But there is no such region. Similarly if I lies in R_3, I_A and I_C also lie in R_3 but I_B in R_5 which is impossible because I_B lies on the internal bisector of B. If I lies in R_4, I_A will lie in R_2 and I_B, I_C in R_4 which is certainly possible. Hence the two rectangular hyperbolas will be real only when I lies in R_4. Also the two branches of the hyperbolas will lie in R_2 and R_4. Hence because the parabola $\sum$ lies in R_3 we have the theorem that "The rectangular hyperbolas belonging to a real four line system of conics do not intersect the inparabola $\sum$ of the system in real points."

REMARKS ON THE PREVIOUS PAPER

BY

A. NARASINGA RAO, *Annamalainagar*

It would be interesting to find out, how far the main results of Mr. Rangachariar's paper may be obtained from considerations of continuity and without the use of equations.

The two conics through a point T are in general distinct, and will coincide only when T lies on one of the given lines. Hence it may be expected that on crossing any of the four lines, the conics should pass over from a real pair to a pair of imaginary conics. This is confirmed by relation (2·2). In R_1 both the conics are real ellipses. Hence the regions in which the conics are real are $R_1\, R_2\, R_3\, R_4\, R_5$ and those in which they are both imaginary are $I_1\, I_2\, I_3\, I_4\, I_5\, I_6$.

Suppose that in passing from T_1 to T_2 one of the two conics changes its character from an ellipse to a hyperbola. By considerations of continuity there must be an intermediate position where this particular conic is a parabola. There being only one parabola in the system, on crossing it one of the two conics through the point will change its character, the other retaining its own.

If there were two parabolas in the system their intersection would provide a point where a change in the character of both the conics may be expected.

Let us call a region elliptic or hyperbolic if both the conics through a point T in the region are real ellipses or both real hyperbolas. If one conic is a real ellipse and the other a real hyperbola,

it may be called a mixed region. Now the character of a region changes when we cross one of the 4 given lines or the parabola Σ. The line at infinity introduces no change of character on crossing it; so that we may simplify the author's subdivision of regions by taking $R_3 + R_5$ as a single region, say R_{35}, similarly $R_2 + R_4 = R_{24}$. Mr. Rangachariar's more elaborate discussion leads to the same result, that the character of R_2 is the same as that of R_4 both being hyperbolic.

The region R_{35} is divided into 5 parts by Σ namely, that within the parabola, the 3 regions outside the parabola adjoining K, R, H respectively; and lastly the two infinite bits in R_3 outside Σ and the bit R_5 considered as one region. Let us call these respectively

$$R_{35\Sigma},\ R_{35K},\ R_{35R},\ R_{35H} \text{ and } R_{35P}$$

Now if we take the finite doubled segment SQ or PR or KH it constitutes a line ellipse. A neighbouring line ellipse cuts the relevent segment in points close to its extremities. Hence a point T close to one of the extremities and lying on any of the three finite segments mentioned above, is an elliptic point. This proves therefore that

$$\left.\begin{matrix} R_1 \\ R_{35K} \\ R_{35H} \end{matrix}\right\} \text{are elliptic regions.}$$

Again the remaining parts of the lines SQ, PR and KH produced infinitely and outside the finite segments constitute a degenerate hyperbola and a neighbouring elongated hyperbola will cut these at points close to the extremities S, Q, P, R, K or H. Thus a point T in SQ produced either beyond S or beyond Q is a hyperbolic point and similarly in the case of the other diagonals. This shows that

$$\left.\begin{matrix} R_{24} \\ R_{35R} \\ R_{35P} \end{matrix}\right\} \text{are hyperbolic regions.}$$

Since we may pass over into $R_{3\Sigma}$ by crossing Σ either from purely elliptic or purely hyperbolic regions, it follows that

$R_{3\Sigma}$ is a mixed region.

I wonder whether any plausible reason could be advanced for the elliptic and hyperbolic regions consisting of separate bits while the mixed region $R_{3\Sigma}$ is simply connected. In what respects do the two ellipses in one of the elliptic regions differ from those in the other regions?

ON MR. NARASINGA RAO'S PROBLEM AND ITS RECIPROCAL

BY

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Problem :—*Given four positive numbers, does there exist a tetrahedron* OABC *whose four faces have for their areas the four given numbers, and if so what are the bounds for the volumes of all such tetrahedra?* (Vide Q. 1757).

In this note I will show that the problem is completely equivalent to the following :—

Given four lengths, does there exist a quadrilateral PQRS in space such that PQ, QR, RS, SP are equal to the given lengths, and what are the bounds for the volumes of all such tetrahedra PQRS? Two formulae which are very important for the proof will first be derived.

Let $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ be the vectors represented by OA, OB, OC and let a, b, c, d be the areas of the faces OBC, OCA, OAB and ABC respectively.

We have

$$[\mathbf{A}_1\ \mathbf{A}_2]+[\mathbf{A}_2\ \mathbf{A}_3]+[\mathbf{A}_3\mathbf{A}_1]=[\mathbf{A}_1-\mathbf{A}_3,\ \mathbf{A}_2-\mathbf{A}_3] \quad \ldots \quad (1{\cdot}1)$$

Now write

$$2\mathbf{A}'_3=[\mathbf{A}_1\ \mathbf{A}_2],\ 2\mathbf{A}'_1=[\mathbf{A}_2\ \mathbf{A}_3],\ 2\mathbf{A}'_2=[\mathbf{A}_3\ \mathbf{A}_1]$$
$$-2\mathbf{A}'_4=[\mathbf{A}_1-\mathbf{A}_3,\ \mathbf{A}_2-\mathbf{A}_3]\ ;$$

then $\mathbf{A}'_1$, $\mathbf{A}'_2$, $\mathbf{A}'_3$, $\mathbf{A}'_4$ will have magnitudes equal to a, b, c, d respectively, and (1·1) becomes

$$\mathbf{A}'_1+\mathbf{A}'_2+\mathbf{A}'_3+\mathbf{A}'_4=0 \quad \ldots \quad (1{\cdot}2)$$

Drawing the quadrilateral PQRS in space such that $PQ=\mathbf{A}'_1$, $QR=\mathbf{A}'_2$ $RS=\mathbf{A}'_3$, we see by (1·2) that $SP=\mathbf{A}'_4$

We see at once that the necessary and sufficient condition for the existence of tetrahedra with four given areas is simply that it must be possible to form a closed quadrilateral PQRSP in space such that $PQ=a$, $QR=b$, etc. The necessary and sufficient condition for the latter is that the sum of any three of the given numbers must be greater than the fourth, i.e. if $a<b<c<d$, then we must have

$$a+b+c>d \quad \ldots \quad (1{\cdot}3)$$

2. If V, $\overline{V}$ be the volumes of the tetrahedra OABC, PQRS, we have

$6V = \mathbf{A}_1 \cdot [\mathbf{A}_2\, \mathbf{A}_3]$ and $6\,\overline{V} = \mathbf{A}'_1 \cdot [\mathbf{A}'_2\, \mathbf{A}'_3]$, as can be easily proved.

Since by a well-known result $2\mathbf{A}'_1 \cdot [2\mathbf{A}'_2, 2\mathbf{A}'_3] = \{\mathbf{A}_1[\mathbf{A}_2\mathbf{A}_3]\}^2$ we have therefore

$$36V^2 = 2^3 \cdot \mathbf{A}'_1 \cdot [\mathbf{A}'_2\, \mathbf{A}'_3] = 2^3 \cdot 6\,\overline{V}$$

$$\text{i.e.} \qquad V^2 = \tfrac{4}{3}\,\overline{V} \qquad \ldots \quad (2{\cdot}1)$$

Now it is easy to see that $\overline{V} < \frac{1}{6}\, PQ{\cdot}QR{\cdot}RS = \frac{1}{6}\, abc$

$$\therefore \qquad V^2 \leqslant 2\,abc/9. \qquad \ldots \quad (2{\cdot}2)$$

The relation (2·1) shows that the tetrahedron OABC of maximum or minimum volume (having the values a, b, c, d for the areas of its four faces) corresponds to the reciprocal tetrahedron PQRS of maximum or minimum volume having for its sides the lengths a, b, c, d. The inequality (2·2) gives an upper bound for the volume of OABC.

Consider now the tetrahedron PQRS. Let $PR = e$ and let the angle between the planes PSR, PQR be ϕ.

Then volume of $PQRS = \overline{V} = \dfrac{2}{3} \cdot \dfrac{\text{area } PQR \times \text{area } PSR}{PR} \cdot \sin\phi$

Now $(\text{area } PQR)^2 = -\frac{1}{16}\,\{e^2 - (a+b)^2\}\{e^2 - (a-b)^2\}$

and $(\text{area } PSR)^2 = -\frac{1}{16}\,\{e^2 - (c+d)^2\}\{e^2 - (c-d)^2\}$

$$\therefore \quad \overline{V}^2 = \frac{4}{9 \cdot 16^2}\left[\frac{\{e^2-(a+b)^2\}\{e^2-(a-b)^2\}\{e^2-(c+d)^2\}\{e^2-(c-d)^2\}}{e^2}\right]\sin^2\phi$$

$$\text{i.e.} \quad \overline{V}^2 = K.\ F(e^2)\sin^2\phi \qquad \ldots \quad (2{\cdot}3)$$

where K is a constant, and $F(e^2)$ denotes the function inside [ ].

Since $(a+b)^2$, $(c+d)^2$ are greater than e^2 and $e^2 > (c-d)^2$, $e^2 > (a-b)^2$, it follows that $F(e^2)$ is bounded. It is at once evident that the lower bound of all the $\overline{V}$'s is zero, since $\sin\phi$ can be made as small as we like. Hence also the lower bound of all the V's will be zero, on account of (2·1).

3. Now the maximum of $\overline{V}$ will correspond to $\sin\phi = 1$. To examine the maxima of $F(e^2)$ in the region

$$\left.\begin{matrix}(a+b)^2\\(c+d)^2\end{matrix}\right\} > e^2 > \left\{\begin{matrix}(a-b)^2\\(c-d)^2,\end{matrix}\right.$$

put $e^2 = t$. Then

$$\frac{F'}{F} = \frac{1}{t-(a+b)^2} + \frac{1}{t-(a-b)^2} + \frac{1}{t-(c+d)^2} + \frac{1}{t-(c-d)^2} - \frac{1}{t},$$

the dash denoting differentiation w.r.t. t.

Suppose the four numbers $(a+b)^2$, $(a-b)^2$, $(c+d)^2$, $(c-d)^2$ to be arranged in order of magnitude and let then $(c+d)^2 \geqslant (a+b)^2 \geqslant (a-b)^2 \geqslant (c-d)^2$.

Drawing the graph of $y = \frac{F'(t)}{F(t)}$, it will be easy to see that the curve crosses the t-axis once in each of the following intervals :—

(i) $t < 0$

(ii) $(c-d)^2 < t < (a-b)^2$

(iii) $(a-b)^2 < t < (a+b)^2$

(iv) $(a+b)^2 < t < (c+d)^2$

so that the equation $\frac{F'(t)}{F(t)} = 0$ when simplified will be a quartic with 4 real roots, of which, since in our case $(a+b)^2 > e^2 > (a-b)^2$, we are concerned only with the root in the interval (iii). Hence

there is a unique maximum tetrahedron in the real field.‡

The relation (2·1) now enables us to conclude that there is only one maximum tetrahedron OABC with four given areas.

Instead of starting with the planes PQR, PSR we could have carried out the above process with the areas PSQ, RSQ and we could prove that the planes PSQ, RSQ are orthogonal in the maximum tetrahedron. The maximum tetrahedron is thus characterised by the property that the planes meeting in a diagonal of the quadrilateral PQRS are orthogonal to each other. Since by our previous argument there is only one real maximum tetrahedron PQRS having specified sides (PQ, QR, RS, SP), we see that given the sides (PQ, QR, RS, SP) and the property of orthogonality of the two pairs of planes meeting in the diagonals PR and SQ, the tetrahedron must be unique.

By a little calculation we can prove that in the maximum reciprocal tetrahedron OABC, opposite edges (OA, BC) etc. will be orthogonal.

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‡ Identical arguments will hold good when $(c+d)^2 \geqslant (a+b)^2 \geqslant (c-d)^2 \geqslant (a-b)^2$ or the other two similar arrangements when $(a+b)^2 > (c+d)^2$

4. *Summary* :—(1) If $a<b<c<d$, then $a+b+c>d$ is the condition for the existence of real tetrahedra having for the areas of the four faces the four given numbers a, b, c, d.

(2) $0 \leqslant V^2 \leqslant \frac{2}{9} abc$.

(3) the problem is equivalent to the problem of finding the upper and lower bounds of all tetrahedra PQRS such that $PQ=a$, $QR=b$, $RS=c$, $SP=d$; and if $\overline{V}$ = volume of PQRS, then $V^2=4\overline{V}/3$.

(4) the tetrahedron having the maximum volume is orthogonal (i.e opposite edges are orthogonal) and an orthogonal tetrahedron having specified areas is unique.

(5) The problem can be easily generalised to n dimensions, with identical results.

In a paper submitted to the Indian Academy of Sciences* I have dealt with the same problem for n dimensions and given different proofs for the above results for n dimensions. Since then I discovered the reciprocal property (3) and the new solution by its use is set forth above.

Note :—Slightly better upper bounds than the one given here in (2) can be found and one such expression is given in my Academy paper. In case $a^2+b^2+c^2=d^2$, it is easy to see that

$$\overline{V} = \frac{1}{6} abc \text{ for the maximum tetrahedron}$$

and hence $V^2 = \frac{2}{9} abc$ for the maximum case.

* *Proc. Ind. Acad. Sc.* Vol. VII, No. 5.

GLEANINGS

Fashions in Mathematics

"Someone strikes gold, and the rush is on until the diggings become less productive; then on to the next rich prospect.

"Let me conclude with another figure of speech perhaps unfortunately borrowed from military science, but in some ways more apt than the comparison I have made to a gold rush. I think of mathematicians as behind lines that separate them from the territory not yet acquired. Some genius marks out a great salient to be added to the conquered regions. Other mathematicians occupy it and push it out still further. Finally it merges with other salients which tend to disappear as such and new wedges are extended from the consolidated territory into regions previously unconquered. These salients correspond to what I have called fashions. They are the landmarks of progress."

From D. R. CURTISS' article *Fashions in Mathematics* quoted in *The Mathematics Teacher*.

NOTES AND DISCUSSIONS

Limiting forms of Carnot's Theorem

1. Newton's theorem states that for fixed directions of the intersecting chords POQ and ROS,

$$(OP \cdot OQ)/(OR \cdot OS) \text{ is constant.} \qquad \dots \quad (1{\cdot}1)$$

When all the lengths involved are finite and different from zero, the theorem can be stated in the form

$$\frac{OP \cdot OQ \cdot O'R' \cdot O'S'}{OR \cdot OS \cdot O'P' \cdot O'Q'} = 1 \qquad \dots \quad (1{\cdot}2)$$

if the chord POQ is parallel to the chord P′O′Q′, and the chord ROS to the chord R′O′S′; and may be proved by taking two Carnot triangles with O, O′ as common vertices and making the other vertices tend towards infinity along the directions POQ and ROS respectively. When some of the lengths involved are infinite or zero, at least one of the Carnot triangles we have to consider will have one vertex on the conic at infinity rendering the direct application of Carnot's theorem impossible and so leaving open any conclusion the value of the compound ratio (1·2) unless it can be obtained from a limiting process. The object of this paper is to obtain limiting forms of Carnot's theorem (1) when none of the vertices lies on the curve, only one vertex is at infinity, and the join of the other vertices meets the curve in one finite point and one point at infinity; (2) when only one vertex is on the curve, and the other two vertices are at a finite distance. The first of these two theorems is really equivalent to Carnot's theorem and so equivalent to Newton's theorem for finite chords while the latter (Theorem 2) appears not to have been noticed before.

THEOREM. *If POQ, P′O′Q′ be two parallel finite chords of a conic and OO′ meets the conic in only one finite point R and a point at infinity ω, then*

$$\frac{OP \cdot OQ}{OR} = \frac{O'P'O'Q'}{O'R}$$

Proof. By applying Carnot's theorem to a finite triangle TOO′, TO and TO′ meeting the conic in P_1, Q_1; P_1', Q_1', respectively and making T tend towards infinity in the direction POQ, we have

$$\lim \frac{TP_1' \cdot TQ_1'}{TP_1 \cdot TQ_1} \cdot \frac{OP_1 \cdot OQ_1}{OR_1 \cdot O\omega} \cdot \frac{O'R_1 \cdot O'\omega}{O'P_1' \cdot O'Q_1'} = 1,$$

$$i.e., \quad \frac{OP \cdot OQ}{O'P' \, O'Q'} = \frac{OR}{O'R}.$$

Corollary. *If POQ is a variable chord of a hyperbola in a fixed direction meeting the conic in two finite points P, Q and an asymptote in O, then $OP \cdot OQ$ is constant.*

This follows at once from the theorem above by noting that the limit (OR/O′R) is unity as the parallel O O′ tends to the asymptote itself.

THEOREM 2. *If a conic passing through the vertex A of a triangle ABC cuts BC in A_1, A_2 and AB, AC in C_1 and B_1 respectively and if BC and the tangent at A meet in α, then*

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1}{AB_1} \cdot \frac{AC_1}{BC_1} = \frac{B\alpha}{C\alpha}.$$

Proof. Let A′ be any point in CA: and BA′ cut the conic in C'_1 and C_2; let AC_2 meet CB in α'. Then by Carnot's theorem applied to the triangle A′BC, we have

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1 \cdot CA}{A'B_1 \cdot A'A} \cdot \frac{A'C_2 \cdot A'C'_1}{BC_2 \cdot BC_1'} = 1.$$

Also $\quad \dfrac{A'A}{CA} \cdot \dfrac{BC_2}{A'C_2} \cdot \dfrac{C\alpha'}{B\alpha'} = 1,$ by Menelau's theorem.

Hence $\quad \dfrac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \dfrac{CB_1}{A'B_1} \cdot \dfrac{A'C_1'}{BC_1'} \cdot \dfrac{C\sigma'}{B\alpha'} = 1,$

by compounding the above two ratios.

The theorem now follows by making A′ tend to A along AC and noting that C_1' and α' tend to C_1 and α respectively.

Corollary 1. *If two chords BA_1, AB_1 of a conic meet in C and the tangents at A and B meet BC and CA in α and β respectively, then*

$$\frac{CB_1}{AB_1} \cdot \frac{BA_1}{CA_1} = \frac{B\alpha}{C\alpha} \cdot \frac{C\beta}{A\beta}.$$

This follows by making B tend to A_1 in the result of the theorem and using Menelau's theorem as above to eliminate the evanescent lengths.

If the conic is a hyperbola, making A and B tend to infinity along the curve and noting that the ratios $B\alpha/BA_1$, $AB_1/A\beta$ tend to unity, we obtain

Cor. 1. $\qquad \dfrac{CA_1}{CB_1} = \dfrac{C\alpha}{C\beta}$, which is well-known.

In the theorem above by making A tend to infinity along the curve, say to the point ω, we get the following result:

Corollary 2. *If BC meets a conic passing through a point at infinity ω in A_1, A_2 and lines $B\omega$, $C\omega$ meet the conic again in finite points C_1 and B_1 respectively and BC meets the tangent at ω in α, then*

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1}{BC_1} = \frac{B\alpha}{C\alpha}.$$

The result of this corollary applied to the hyperbola brings into evidence the mistake that gave rise to fallacy in Suryanarayana Iyer's example. His argument amounts to replacing $B\alpha/C\alpha$ by unity which certainly is not justifiable unless α is at infinity.

Corollary 3. *If POQ $P'O'Q'$ be finite chords of a hyperbola meeting one of the asymptotes in r and r' respectively and lines ωOR, $\omega O'R'$ (ω being the point at infinity on the asymptote) meet the conic in R and respectively then*

$$\frac{OP \cdot OQ}{OR \cdot Or} = \frac{O'P' \cdot O'Q'}{O'R' \cdot O'r'}.$$

Proof. Let OR and P′Q′ meet in O″. Then by theorem 1, we have

$$\frac{OP \cdot OQ}{OR} = \frac{O''P' \quad O''Q'}{O''R} \qquad \ldots \quad (1\cdot 3)$$

and again by theorem 2, Cor. 2, we have

$$\frac{O'P' \cdot O'Q'}{O''P' \cdot O''Q'} \cdot \frac{O''R}{O'R'} = \frac{O'r'}{O''r'} = \frac{O'r'}{Or}.$$

$$\text{Hence } \frac{O'P' \cdot O'Q'}{O'R' \cdot O'r'} = \frac{O''P' \cdot O''Q'}{O''R \cdot Or} = \frac{OP \cdot OQ}{OR \cdot Or}, \text{ by} \qquad \ldots \quad (1\cdot 3).$$

I wish to express my thankfulness to Dr. V. Ramaswami for suggesting the line of enquiry followed in this paper.

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N. RAMA RAO.

The α and β Hyperboloids

It is the object of this note to specify the conicoids in space which are analogous to rectangular hyperbolas in the plane. Schroter* has discussed equilateral and orthogonal hyperboloids; equilateral hyperboloids are these which possess an infinite number of trirectangular generators; orthogonal hyperboloids are these in which two pairs of generators are perpendicular to the planes of circular sections. The hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is equilateral if $\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}=0$ and orthogonal if $\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}=0$. The former constitutes an analogue of the rectangular hyperbola in the plane while the latter does not.

To my mind, there are two analogues, one the equilateral hyperboloid and the other in which the director sphere is a point-sphere or the asymptotic cone has an infinite number of trirectangular tangent planes†. These correspond to quadrics whose linear and quadratic invariants vanish. I have ventured to name them as the α and β hyperboloids respectively. If $ax^2+by^2+cz^2+2hxy+2gzx+2fyz+d=0$ be the general equation in rectangular Cartesian co-ordinates, $a+b+c=0$ is the condition for the α hyperboloid and $bc-f^2+ca-g^2+ab-h^2=0$ is the condition for the β hyperboloid. The α and β hyperboloids are mutually reciprocal with respect to the centre and may be denoted by H_α and H_β.

2. The centres of H_α's self-polar w. r. t. a given tetrahedron ABCD is the cubic Surface‡ which is the locus of the point, the feet of the perpendiculars from which on the four faces are coplanar.

If $ax^2+by^2+cz^2+dw^2=0$ be the equation of the conicoid, then the condition that the sum of the reciprocals of the squares of the axes is zero is §

$$aA_1^2+b\cdot B_1^2+c\cdot C_1^2+d\cdot D_1^2=0$$

where A_1 B_1 C_1 D_1 are the areas of the four faces.

The centre which is given by

$$ax=by=cz=dw$$

lies on the surface

$$\frac{A_1^2}{x}+\frac{B_1^2}{y}+\frac{C_1^2}{z}+\frac{D_1^2}{w}=0.$$

* Quoted in 'Salmon and Rogers' "*Solid Geometry*" 5th Ed., P. 116.

† These are called rectangular hyperboloids, see Smith's *Solid Geometry*. 1907 P. 131. Ex. 1.

‡ See Wilkinson *J. I. M. S.* Vol. V. P. 53.

§ See Wilkinson *J. I. M. S.* Vol. V. P. 208.

3. All Conicoids passing through the centres of the eight tangent spheres of a tetrahedron are α-hyperboloids.

It is easy to verify that all conicoids through 7 out of the 8 points like $(1, 1, 1, 1)(-1, 1, 1, 1)$ etc. pass also through the eighth and that the equation of the conicoid is $p\alpha^2+q\beta^2+r\cdot\gamma^2+s\cdot\delta^2=0$ (using normal co-ordinates) with the condition $p+q+r+s=0$. This corresponds to the condition $aA_1{}^2+bB_1{}^2+cC_1{}^2+dD_1{}^2=0$ in areal co-ordinates. The centres of the conicoids lie on cubic surface referred to in para 2 supra. Knowing that the surface of centres of quadrics through 8 points passes through the mid-points of the 28 joins of the 8 points, we deduce that the Cubic Surface corresponds to the circumcircle in the plane, which is the centre-locus of rectangular hyperbolas passing through the in and ex-centres of the fundamental triangle.

4. The centres of β hyperboloids self polar to the fundamental tetrahedron lie on the circumsphere.

The condition that $ax^2+by^2+bz^2+dw^2=0$ is an β-hyperboloid

$$\sum ab\cdot CD^2=0.$$

The centre which is given by $ax=by=cz=dw$ lies on $\Sigma zwCD^2=0$ which is the circumsphere*.

5. All conicoids passing through 5 mutually orthocentric points are α hyperboloids. This can be proved by using the condition for α hyperboloids and the relations among the edges.

6. A family of Confocal conicoids contains two α members and one β member.

7. The centres of all β hyperboloids touching six given planes is a sphere†.

Madras. } N. DURAIRAJAN.

* See Salmon and Rogers *Solid Geometry* P. 215. Ex. 2

† See Smith's *Solid Geometry*. P. 137. Ex. 4.

A Note on the Curvature of Plane Curves

1. The object of this note is to obtain by analytical methods certain important limits relating to the curvature of plane curves. We take the origin at a point P of the given curve, and the tangent and normal at P as co-ordinate axes. The equation of the curve may then be written in the form

$$y=f(x)=a_2x^2+a_3x^3+\cdots\cdots\cdots$$

(exceptions may occur where it is not possible to expand $f(x)$ in a series of ascending powers of x).

2. THEOREM I. *If C is the centre of curvature at P, while Q is a neighbouring point on the curve, then as Q tends to P, the ratio $(CQ-CP)/PQ^3$ tends to a definite limit.*

The co-ordinates of the point C are easily seen to be $(0, 1/2a_2)$.

$$\therefore \frac{CQ-CP}{PQ^3}=\frac{\{x^2+(y-1/2a_2)^2\}^{\frac{1}{2}}-1/2a_2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$=\frac{\{x^2+y^2-y/a_2+1/4a_2^2\}^{\frac{1}{2}}-1/2a_2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$=\frac{\left\{x^2+(a_2^2x^4+\cdots)-(x^2+\frac{a_3}{a_2}x^3+\cdots)+\frac{1}{4a_2^2}\right\}^{\frac{1}{2}}-\frac{1}{2a_2}}{(x^2+a_2^2x^4+\cdots\cdots)^{\frac{3}{2}}}$$

$$=\frac{\left\{\frac{1}{4a_2^2}-\frac{a_3}{a_2}x^3+\cdots\cdots\cdots\right\}^{\frac{1}{2}}-\frac{1}{2a_2}}{x^3(1+a_2^2x^2+\cdots\cdots)^{\frac{3}{2}}}$$

$$=\frac{\frac{1}{2a_2}(1-4a_2a_3x^3+\cdots\cdots\cdots)^{\frac{1}{2}}-\frac{1}{2a_2}}{x^3(1+a_2^2x^2+\cdots\cdots\cdots)^{\frac{3}{2}}}$$

$$=\frac{\frac{1}{2a_2}(1-2a_2a_3x^3+\text{higher powers})-\frac{1}{2a_2}}{x^3+\text{higher powers}}$$

which tends to $-a_3$ as $x\to 0$.

3. THEOREM II. *If P is a point on a curve and QR a chord parallel to the tangent at P, N the middle point of QR; then if Q and R both tend to P in such a manner that QR moves parallel to itself, the equation of PN will ultimately be**

$$Y\left(\frac{d\rho}{ds}\right)_0=3X.$$

* Vide Joseph Edwards: Differential Calculus p. 110 Question 38.

Let the co-ordinates of Q and R be $\{h_1, f(h_1)\}$ and $\{h_2, f(h_2)\}$ respectively. Since QR is parallel to the axis of x, its slope is

$$\frac{f(h_2)-f(h_1)}{h_2-h_1}=\frac{(a_2h_2^2+\cdot\cdot\cdot)-(a_2h_1^2+\cdot\cdot\cdot)}{h_2-h_1}=0,$$

or $$a_2(h_1+h_2)+a_3(h_1^2+h_1h_2+h_2^2)+\cdot\cdot\cdot\cdot\cdot=0 \qquad \ldots \quad (3\cdot1)$$

Also the slope of PN is

$$\tan\ \theta=\frac{a_2(h_1^2+h_2^2)+a_3(h_1^3+h_2^3)+\cdot\cdot\cdot\cdot}{h_1+h_2}$$

Using (3·1) we get

$$\tan\ \theta=\frac{a_2(h_1^2+h_2^2+a_3(h_1^3+h_2^3)\ +\ \cdot\cdot\cdot\cdot}{-\ \dfrac{a_3}{a_2}\ (h_1^2+h_1h_2+h_2^2)-\cdot\cdot\cdot\cdot}$$

As Q and R approach P subject to the condition that $h_1+h_2=0$, or $h_1=-h_2$, $\tan\ \theta\rightarrow\ -\dfrac{2a_2^2}{a_3}$.

Now $\rho=(1+y_1^2)^{\frac{3}{2}}/y_2=1/y_2$ at the point P, (0, 0)

$$\therefore\quad \left(\frac{d\rho}{ds}\right)_0=\left(\frac{d\rho}{dx}\right)_0=-y_3/y_2^2=-3a_3/2a_2^2 \text{ since } \left(\frac{ds}{dx}\right)_0=1.$$

Hence the equation of PN ultimately takes the form

$$Y\left(\frac{d\rho}{ds}\right)_0=3X.$$

4. THEOREM III. * *If P, Q, R be three points on a curve and p, q, r the points of intersection of the three pairs of tangents at Q, R; R, P; P, Q respectively, then as Q and R both tend to P*

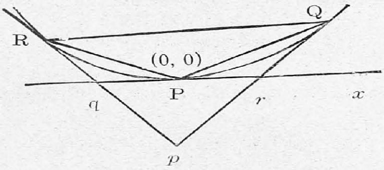
$$\text{(A)}\qquad \frac{qr}{QR}\rightarrow\frac{1}{2};\quad \frac{rp}{RP}\rightarrow\frac{1}{2};\quad \frac{pq}{PQ}\rightarrow\frac{1}{2};$$

$$\text{(B)}\qquad \frac{\triangle pqr}{\triangle PQR}\rightarrow\frac{1}{2}.$$

The equation to the tangent at Q is

$$Y-f(h_1)=f'(h_1)\ (X-h_1).$$

Hence at r where it meets $Y=0$, we have



$$X=h_1-\frac{f(h_1)}{f'(h_1)}=h_1-\frac{a_2h_1^2+\ \cdot\cdot\cdot\cdot}{2a_2h_1+3a_3h_1^2+\cdot\cdot}$$

$$\therefore\quad Pr=\frac{a_2h_1+\ \cdot\cdot\cdot\cdot\cdot\cdot}{2a_2+3a_3h_1+\ \cdot\cdot\cdot}=\frac{h_1}{2}+O(h_1^2)$$

Similarly $$Pq=\frac{h_2}{2}+O(h_2^2)$$

* These properties were communicated to me by Dr. Chowla.

$$\therefore \quad qr = \tfrac{1}{2}(h_1 - h_2) + O(h_1^2) + O(h_2^2) \qquad \ldots \quad (4\cdot 1)$$

Also $QR = \sqrt{(h_1 - h_2)^2 + \{f(h_1) - f(h_2)\}^2}$

$$= \sqrt{(h_1 - h_2)^2 + \{a_2(h_1^2 - h_2^2) + a_3(h_1^3 - h_2^3) + \cdots\}^2}$$

$$= (h_1 - h_2)\sqrt{1 + \{a_2(h_1 + h_2) + \cdots\}^2}$$

$$= h_1 - h_2 + \text{higher powers of } h_1 \text{ and } h_2 \qquad \ldots \quad (4\cdot 2)$$

From (4·1) and (4·2) $\dfrac{qr}{QR} \to \dfrac{1}{2}$ as $h_1, h_2 \to 0$.

$$\triangle PQR = \tfrac{1}{2}\begin{vmatrix} 0, & 0, & 1 \\ h_1, & f(h_1), & 1 \\ h_2, & f(h_2), & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}\{h_1 f(h_2) - h_2 f(h_1)\}$$

$$= \tfrac{1}{2}\{h_1(a_2 h_2^2 + a_3 h_2^3 + \cdots) - h_2(a_2 h_1^2 + a_3 h_1^3 + \cdots)\}$$

$$= \tfrac{1}{2} h_1 h_2 \{a_2(h_2 - h_1) + a_3(h_2^2 - h_1^2) + \cdots\}$$

$$= \tfrac{1}{2} a_2 h_1 h_2 (h_2 - h_1) + \text{higher powers of } h_1 \text{ and } h_2 \qquad \ldots \quad (4\cdot 3)$$

Where tangents at Q and R meet, we have

$$\frac{Y - f(h_1)}{f'(h_1)} - \frac{Y - f(h_2)}{f'(h_2)} = h_2 - h_1$$

or $$\frac{Y - a_2 h_1^2 - \cdots}{2a_2 h_1 + 3a_3 h_1^2 + \cdots} - \frac{Y - a_2 h_2^2 - \cdots}{2a_2 h_2 + 3a_3 h_2^2 + \cdots} = h_2 - h_1$$

or $$\left(\frac{Y}{2a_2 h_1} + \cdots - \frac{1}{2} h_1 + \cdots\right) - \left(\frac{Y}{2a_2 h_2} + \cdots - \frac{1}{2} h_2 + \cdots\right) = h_2 - h_1$$

or $$\frac{Y}{2a_2} \cdot \frac{h_2 - h_1}{h_1 h_2} + \tfrac{1}{2}(h_2 - h_1) + \cdots = h_2 - h_1$$

whence $\quad Y = a_2 h_1 h_2 +$ terms of higher order.

$$\triangle pqr = \tfrac{1}{2} qr . (-Y) = -\tfrac{1}{2}\{\tfrac{1}{2}(h_1 - h_2) + \cdots\}\{a_2 h_1 h_2 + \cdots\}$$

$$= \tfrac{1}{4} a_2 h_1 h_2 (h_2 - h_1) + \text{higher powers of } h_1 \text{ and } h_2 \qquad \ldots \quad (4\cdot 4)$$

From (4·3) and (4·4) $\dfrac{\triangle pqr}{\triangle PQR} \to \dfrac{1}{2}$ as $h_1, h_2 \to 0$.

Cor. *The circumradius of the triangle pqr is $\rho/4$, ρ being the radius of curvature at P.*

For the circumradius of $\triangle pqr = \dfrac{pq \cdot qr \cdot rp}{4 \triangle pqr} = \dfrac{\frac{1}{2} PQ \cdot \frac{1}{2} QR \cdot \frac{1}{2} RP}{4 \cdot \frac{1}{2} \triangle PQR}$

$$= \frac{1}{16} \cdot \frac{PQ \cdot QR \cdot RP}{\triangle PQR} = \frac{\rho}{4}.$$

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A Note on Wilson's Quotient

1. Menon[1] has recently proved that

$$\sum_{i=1}^{p-1} i^{p-1} \equiv (p-1)! + p \pmod{p^2},\ p \text{ being an odd prime.} \qquad (1)$$

In this note, I deduce the above result from the following two results proved by me elsewhere[2].

$$\sum_{i=1}^{n} i^s = \sum_{j=1}^{s} \left\{ G(-j-1,\ s-j).\ \frac{(n+1)^{(j+1)}}{j+1} \right\},\ s>0; \qquad (A)$$

where $n^{(k)} = n(n-1)(n-2)(n-3)\ldots(n-k+1)$;

and $G(j,\ r) \equiv G(p+j,\ r) \pmod{p},\ 0<r\leqslant p-2.$ (B)

It is then shown that[3]

$$(p-1)! \equiv p\,(B_{p-1}-1) \pmod{p^2}, \qquad (2)$$

where B_{p-1} denotes as usual a number of Bernoulli.

2. *Proof of* (1). In (A), let $n=s=p-1$; then

$$S = \sum_{i=1}^{p-1} i^{p-1} = \sum_{j=1}^{p-1} \left\{ G(-j-1,\ p-j-1).\ \frac{p^{(j+1)}}{j+1} \right\},$$

$$= (p-1)! + \sum_{j=2}^{p-1} \left\{ G(-j,\ p-j).\ \frac{p^{(j)}}{j} \right\};$$

Hence
$$\frac{S-(p-1)!}{p} = \sum_{j=2}^{p-1} \left\{ G(-j,\ p-j).\ \frac{(p-1)^{(j-1)}}{j} \right\},$$

$$\equiv \sum_{j=2}^{p-1} \left\{ G(p-j,\ p-j).\ \frac{(p-1)^{(j-1)}}{j} \right\} \pmod{p},$$

$$\equiv (p-1)! \left\{ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{p-1} \right\} \pmod{p},$$

$$\equiv 1 \pmod{p}.$$

This proves the theorem.

3. *Proof of* (2). We have

$$\sum_{i=1}^{n} i^s = \frac{n^{s+1}}{s+1} + \left\{ \frac{1}{2} + \frac{B_2}{2!} D + \frac{B_4}{4!} D^3 + \frac{B_6}{6!} D^5 + \cdots \right\} n^s.$$

Hence $S = \sum_{i=1}^{p} i^{p-1} - p^{p-1}$,

$$= \left[\left\{ \frac{1}{2} + \frac{B_2}{2!} D + \frac{B_4}{4!} D^3 + \cdots + \frac{B_{p-1}}{(p-1)!} D^{p-2} \right\} n^{p-1} \right]_{n-p},$$

$$\equiv B_{p-1} \cdot p \pmod{p^2}.$$

Therefore $(p-1)! \equiv p\,(B_{p-1} - 1) \pmod{p^2}$.

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1. A Theorem on Congruence. *Jour. Ind. Math. Soc.*, 2, (1936-37), 332-333.
2. On the Numbers of Ward and Bernoulli. *Proc. Indian Acad. Sci.*, 3, (1936), 193-200. Congruence Properties of G-Functions. *Proc. Edin. Math. Soc.*, 4, (1935), 61-66.
3. This result is due to Glaisher and Lerch. See E. Lehmer's note in the *American Math. Monthly* 44, April (1937).

ASTRONOMICAL NOTES

Planet Notes from May to September 1938

Note:—All times mentioned here are Indian Standard Time and the co-ordinates and planets given are for 5 hours 30 minutes I. S. T.

Sun.—During the period May to September the Sun will be passing from the centre of Aries to the centre of Virgo. Its Right Ascension will be changing from 2 h. 30 m. to 12 h. 19 m. and its Declination being North increases from 14° 48′ and attains its maximum 23° 27′ on June 22nd when the Sun will be at the Summer solstice. After this date the declination again falls to 0° 0′ on September 23rd when the Sun will be at the Autumnal Equinox and then the declination changes South. On 29th May there will be a total eclipse of the Sun invisible in India.

The Sun's position at the beginning of each month is as follows :—

	May	*June*	*July*	*August*	*September.*
R. A.	2 h 30 m	4 h 33 m	6 h 37 m	8 h 42 m	10 h 38 m
Dec.	14° 48′	21° 56′N	23° 10′N	18° 15′N	8° 37′N

Moon.—The following phases of the Moon will occur during the period from May to September.

	New Moon	*1st Quarter*	*Full Moon*	*Last Quarter*
May	29 at 7–30 P. M.	7 at 2–54 A. M.	14 at 2–9 P. M.	22 at 6–6 P. M.
June	28 at 2–40 A. M.	5 at 10–2 A. M.	13 at 5–17 A. M.	21 at 7–22 A. M.
July	27 at 9–24 A. M.	4 at 7–17 P. M.	12 at 8–35 P. M.	20 at 5–49 P. M.
Aug.	25 at 4–47 P. M.	3 at 7–30 A. M.	11 at 11–27 A. M.	19 at 2–0 A. M.
Sep.	24 at 2–4 A. M.	1 at 10–58 P. M.	10 at 1–38 A. M.	17 at 8–42 A. M.

The Moon will be nearest the Earth on 2nd, 30th May, 28th June., 26th July, 23rd August and 20th September. It will be farthest from the Earth on 18th May, 14th June, 12th July, 8th August and 4th September. On 14th May there will be a total eclipse of the Moon which will be invisible in India.

Moon's position at the beginning of each month:—

	May	*June*	*July*	*August*	*September.*
R. A.	3 h 11 m	6 h 58 m	9 h 33 m	12 h 44 m	15 h 51 m
Dec.	18° 26′ N.	18° 53′ N.	9° 8′ N.	8° 15′ S.	19° 20′ S.

Mercury.—The path of the planet is from Aries to Leo during this period. During the month of May and September the planet will appear as a morning object rising an hour before the Sun and during the latter part of July and early part of August it will appear as an evening object and at other times it is very near the Sun. Mercury will be in conjunction with Uranus on 29th May, with Regulus on 26th July, with Mars on 4th and 16th September and with Neptune on 26th September. It will be in conjunction with the Moon on 28th May and 29th July.

The following phenomena of Mercury will also take place during this period.

Mercury Stationary 4th May, 6th September, 13th August.

,, Greatest elongation West on 19th May and 13th September.

,, ,, East on 31st July.

,, Inferior conjunction with the Sun on 28th August.

The positions of the planet will be as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	1 h 38 m	3 h 6 m	7 h 19 m	10 h 26 m	10 h 9 m
Dec.	9° 5′ N.	14° 49′ N.	24° 2′ N.	8° 17′ N.	7° 55′ N.

Venus.—During the whole of this period Venus will appear as an evening object setting rather late day by day and is very brilliant. The planet passes from Taurus to Libra in the interval. The planet will be in conjunction with the Moon on 2nd, 31st May, 30th June, 30th July, 29th August, 27th September and will be in conjunction with Mars, Neptune and Spica on 8th May, 31st July and 31st August respectively. It will be at its greatest elongation East on 11th September.

The positions of the planet will be as follows:

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	3 h 56 m	6 h 39 m	9 h 10 m	11 h 25 m	13 h 26 m
Dec.	20° 50′ N.	24° 44′ N.	18° 13′ N.	4° 27′ N.	10° 57′ S.

Mars.—The planet moves with a direct motion from Taurus to Leo. During the early part of May the planet will appear as an evening object and then it is too near the Sun to be visible and by the end of September it will be visible in the morning. The planet will be in conjunction with the Moon on the 2nd May, 30th May, 22nd September. It will also be in conjunction with the Sun on 24th July and with Regulus on 5th September.

The positions of the planet will be as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	4 h 12 m	5 h 42 m	7 h 9 m	8 h 34 m	9 h 55 m
Dec.	21° 45′ N.	24° 14′ N.	23° 31′ N.	19° 53 N.	14° 4′ N.

Jupiter.—This planet will be in Aquarius throughout the period. During May it will be seen in the Eastern sky after midnight and it rises earlier day by day and during September the planet will be seen throughout the night. The planet will be in conjunction with the Moon on 22nd May, 19th June, 16th July, 12th August and 8th September. It will be stationary on 22nd June and in opposition with the Sun on 21st August.

The positions of the planet is as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	22 h 3 m	22 h 16 m	22 h 18 m	22 h 9 m	21 h 54 m
Dec.	12° 47′ S.	11° 49′ S.	11° 41′ S.	12° 38′ S.	14° 3′ S.

Saturn.—This planet will be seen early in the morning during May and rising earlier day by day will appear as an object visible throughout the night during September and can be seen on the Eastern sky shortly after sunset. The planet will be in Pisces throughout the period. Saturn will be in conjunction with the Moon on 26th May, 22nd June, 20th July, 16th August, 12th September and will be stationary on 1st August.

The positions of the planet will be as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	0 h 47 m	0 h 59 m	1 h 7 m	1 h 10 m	1 h 8m.
Dec.	2° 42′ N.	3° 52′ N.	4° 34′ N.	4° 43′ N.	4° 17′ N.

Uranus.—Since May the planet will be visible as a morning object. The motion of the planet will be direct. Uranus will be in conjunction with the Sun on 4th May and will be stationary on 24th August.

The positions of the planet will be as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	2 h 45 m	2 h 51 m	2 h 57 m	3 h 1 m	3 h 2m
Dec.	15° 32′ N.	16° 3′ N.	16° 28′ N.	16° 44′ N.	16° 47′ N.

Neptune.—This planet will be visible very near the meridian by 8 o'clock in the night during May and by September it will be too near the Sun to be seen. The planet will be stationary on 22nd June and will be in conjunction with the Sun on 14th September.

The positions of the planet will be as follows:—

	1st May	*1st June*	*1st July*	*1st August*	*1st September.*
R. A.	11 h 20 m	11 h 19 m	11 h 20 m	11 h 23 m	11 h 26 m
Dec.	5° 34′ N.	5° 38′ N.	5° 31′ N.	5° 14′ N.	4° 49′ N.

H. SUBRAMANIA AIYER.

REVIEWS

A. DENJOY: *Theorie des Functions de variables reelles*, 2 parts.

(Actualites Scientifiques et Industrielles, Nos. 451 and 452, price 12 Frs. each)

In these two volumes, the author has made a successful attempt to marshal together the fundamental notions and principles underlying the theory of functions of a real variable. As such, these two themselves constitute a review of the development of the theory from its earliest stages to its present state. In Chapter I, the author traces the development of the notion of a "function" to its present general form. The beginnings of the modern theory of functions could be traced to Abel and Cauchy who were the first to insist on rigorous demonstration in analysis on the lines of Euclid's Elements. But Cauchy's notion of a function did not go beyond analytic functions and it was Riemann who first formulated the general definition of the notion of a function that if to every value of x in an interval (a, b) a definite value y (by some specified law) is associated, then y is said to be a function of x. This was suggested to Riemann by Fourier's work wherein the self-same trigonometric series was found to represent a function whose values, according to the old order of ideas, were totally disconnected in different parts of its interval of definition. Further impetus to the study of the general function was given by Cantor who developed the general theory of sets of points (or assemblages). The pioneers in this field were Borel, Lebesque and Baire. In Chapters I and II of Part I, a short account of these developments is given while in Chapter III, the classification of functions by means of the notions of continuity, convergence and derivability is treated at some length.

In Chapter I (Part II), a short account of the notion of integrals according to Riemann, Lebesque, Stieltjes and Denjoy is given and the relation of these to differentiation is indicated. Chapter II (part II) deals with trigonometric series and the theory of orthogonal functions. A short account of quasi-analytic functions (which like analytic functions are determined by their values at a portion of their interval of definition) is given in Chapter III while the last chapter treats of the modern theory of functionals and functions of assemblages. These latter are functions defined over assemblages whose elements themselves are functions with specified properties. These form special cases of the general theory of abstract spaces and their mutual correspondences which has been developed to a great extent by Hilbert, Frechet and Morse.

V. G. IYER.

B. B. BAGI: *The Analytical Geometry of Conic Sections*, Dharwar, published by the author, Pp. viii + 248, 1936.

This is a book by an author who has already two other useful text-books on Algebra and Plane Trigonometry to his credit. After an introductory chapter discussing the convention regarding the signs of lengths, areas, and angles, the author proceeds to deal with the straight line, circle, parabola, ellipse, and hyperbola referred to rectangular or oblique axes. A detailed discussion of the general equation of the second degree in three chapters enables the author to avoid repetitions in dealing with different species of conics. One chapter is devoted to the polar equation of the line and the conic referred to a focus for pole. Imaginary points and lines are introduced and some elementary results given, though the subject is not pursued further. The book should prove useful as a text-book for the B. A. and B. Sc. Pass courses in Indian Universities. Its value would be further enhanced if the number of problems at the ends of the chapters were increased and graded into two or three classes.

Having dealt with Mr. Bagi's book, the reviewer feels free to indulge his grouse against the teaching of Analytical Geometry and the text-books written therefor. Every mathematical discipline has its own distinctive features which are largely determined by the methods of investigation it employs. There are types of problems which it would seem to invite for study; and there are results for which it could hardly be coaxed to give a laboured verification. Is it unreasonable to demand that the student should have at the end of his graduate course, a fair notion of the most striking results that the new method has achieved and the type of investigation for which it is specially suited? As taught today, the subject is introduced to the student somewhat as follows:—"This is Mr. Cartesian Geometry who can do almost everything that Euclid did. He has a panacea for all geometrical troubles, which like all panaceas will sometimes act quickly and at others with alarming slowness, if at all. He has a reputation for dealing with loci, though often the net result of his treatment is a complicated equation which leaves you puzzled as to what it could possibly signify."

How much better would it be if they said instead:—"This is the famous Magician Cartesian Geometry. He has a magic carpet which he calls 'Z' which will carry you to the Land of the Far Away where you will see for yourself those rebellious parallels which would not meet however far they were produced, by Euclid behaving quite as meekly as any other pair of lines. And he has another carpet called '*i*' which will carry you to the Land of the Unseen, a Ghostly Land wherein are to be found the spirits of a pair of points or lines when they suddenly disappear from the Real World locked in each other's arms. And if you should spread both the carpets, why, who can prevent you from reaching even up to those miraculous twin shrines which stand right at the very borders of the Metrical Domain! You are indeed lucky to have Cartesian as your travelling companion."

A. NARASINGA RAO.

ELDERTON, W. P.: *Frequency Curves and Correlation*, Cambridge University Press: 1938. pp. 271; 12/6 nett. (Third Edition)

The amount of original work done in the various branches of mathematical statistics has been steadily increasing during the last thirty years thanks to Professor R. A. Fisher and the late Professor Karl Pearson in England and a number of others in Europe and America. It is becoming more and more difficult to form an idea of the advances made in various directions in the subject and the need for books like the one under review is being increasingly felt by workers in the field.

Mr. Elderton's "Frequency Curves and Correlation" will serve as a good introduction to the work of Karl Pearson and his school—to which the author himself belongs—in the field of frequency curves, correlation, and sampling. The first chapter in the book sets out the task of the statistician and warns him against certain pitfalls on his way. The second introduces the mean, mode standard deviation, and skewness. Then follows an exposition of the method of moments with a detailed discussion of Pearson's system of frequency curves and a comparison of this with the systems associated with Gram, Charlier, and Edgeworth by means of several examples. The various ways of measuring correlation including contingency are explained in Chapters VII—IX and XII—XIII, those in between being devoted to standard errors and the test of goodness of fit. These chapters have been rewritten so as to bring the matter up-to-date and this constitutes the main improvement over the last edition. Most of the numerous appendices are intended for the reader with a high mathematical equipment who wants to go into details. The appendix on the methods of curve fitting including Prof. Fisher's method of maximum likelihood demands a very careful study. One appendix suggests an abridged reading for the non-mathematical reader who wants only a general idea of the subject. The book also contains a large number of tables for reference while working out examples.

Now I venture a few suggestions for the consideration of the author:—

In the chapter on Frequency Distributions, on page 9, in the paragraph on standard deviation we find the statement ". . . . *if a given distribution is arranged in two series, first according to years of age and then in quinquennial age groups, the standard deviation will be five times as large in the latter case as in the former*". This is likely to be misunderstood by the beginner for whom this chapter is intended. What the author means is, as can be guessed from the integrals he gives, that, if the class interval becomes five times as large, the frequencies remaining the same, the standard deviation turns out to be five times as great as before.

The chapter on the Test of Goodness of Fit does not fit well into its present position. Its proper place seems to be between chapters V and VI.

Lastly there are a few misprints here and there but they are of a very minor character.

The book is written in a lucid style which, matter-of-fact as it is, makes reading easy and never tiresome. As regards the general get-up of the volume I have only to remark that the high traditions of the Cambridge University Press have been well kept up.

Annamalainagar. S. SUBRAMANIAN.

BOOKS RECEIVED FOR REVIEW

Actualités Scientifiques et Industrielles, Herman & Cie, Paris.

502. M. CURIE AND M. PROST: *Nécessaire Mathématique.*

542. J. YVON: *Fluctuations en densité.*

543. J. YVON: *La propagation et la diffusion de la lumière.*

544. Y. ROCARD: *Les phenomènes d'autooscillation dans les installations hydrauliques.*

551. A. WEIL: *Sur les espaces a structure uniform et sur la Topologie generale.*

RAMA SASTRY, M. P. *Analytical Geometry for Beginners.* The Ottrumai Office, Saidapet, Madras. Pp. 130; Price Rs. 1-8-0.

SMITH, David Eugene: *Portraits of Eminent Mathematicians, Portfolio II,* Scripta Mathematica, New York. Price $ 3.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been enrolled as members of the Society :—

The Hon'ble Dr. Sir Shah Mohammed Sulaiman, Kt., D.SC, LL. D., Judge, Federal Court, Delhi.

Dr. F. W. Levi, Hardinge, Professor of Mathematics, Calcutta University.

Dr. A. C. Banerji, M.A. (Cantab.), I.E.S., Head of the Department of Mathematics, Allahabad University.

Dr. Avadesh Narain Singh, D.SC., Reader, Lucknow University.

Prof. A. M. Kureishy, M.A, Reader in Mathematics and Provost, Muslim University, Aligarh.

Prof. P. Samuels Lall, M.A. (Princeton), Prof. of Astronomy, Punjab University, Lahore.

Prof. J. A. Strang, M.A., B.SC., Head of the Mathematics Department, Lucknow University.

D. R. Kaprekar Esq., B.SC., High School Teacher, Devlali.

M. R. Yardi Esq., B.A. (Hons.), Bhamburda, Poona.

Dr. N. R. Sen, Ghosh, Professor of Applied Mathematics, Calcutta University.

M. S. V. Raman Esq., B.A. (Hons), Office of the Director of Public, Health, Trichur.

Prof. K. Sambasiva Rao, Department of Mathematics, Andhra University, Waltair.

Prof. M. Raghavacharyulu, M.SC., S.R.R. College, Bezwada.

We learn that the Osmania University Council has approved of the proposal to invite the Indian Mathematical Society to hold its next conference in Hyderabad (Deccan) in the third week of December 1939.

Prof. Dr. F. W. Levi has been elected a member of the Managing Committee of the Indian Mathematical Society in place of the late Dr. S. Mukhopadhyaya.

Prof. B. S. Madhava Rao, of the Engineering College, Bangalore, has been awarded the D.Sc. Degree of the Calcutta University. Prof. Madhava Rao's thesis consists of papers dealing with Born's Field Theory.

Dr. B. Ramamurti and Dr. S. Sivasankaranarayana Pillai, both of the Annamalai University, have been awarded the Ramanujam Memorial Prize of the Madras University.

Prof. R. D. Misra of the Lucknow University has been granted study leave for two years and is working at Edinburgh under Prof. E. T. Whittaker. Prof. R. S. Sarma of the Christ Church College, Cawnpore, has been appointed temporary lecturer for two years in his place.

Dr. V. Ganapati Iyer, D. Sc. (Madras), has been appointed Lecturer in Mathematics in the Hindu College, Masulipatam.

Mr. V. Seetharaman, M. Sc. (Annamalai), has been appointed to a temporary vacancy in the Wilson College, Bombay.

Prof. E. A. Milne has been elected President of the London Mathematical Society.

Frank Morley who was head of the Department of Mathematics at the John Hopkins University since 1900, and retired as Emeritus Professor in 1928, died in October 1937. His name will be familiar to students as the joint author with Harkness of two well-known Treatises on the Theory of Functions. In collaboration with his son Frank V. Morley, he published a stimulating treatise on "Inversive Geometry."

Edmund Landau, formerly of the University of Gottingen, one of the foremost authorities in the Theory of Numbers and the author of several standard works on Analysis and on Number Theory, died on the 19th February 1938 at the age of 62 years.

SOLUTIONS TO QUESTIONS

Question 1673

(V. RAMASWAMY AIYER):—A triangle $\alpha\beta\gamma$ moves rigidly and without rotation, so as to be constantly in perspective with a given triangle $A_0 B_0 C_0$. Prove that there is an infinity of fixed triangles ABC with respect to each of which the moving triangle $\alpha\beta\gamma$ is constantly in perspective. Prove further that the locus of the centre of perspective corresponding to different triangles, forms a system of concentric, similar and similarly situated conics (briefly *co-asymptotic conics*).

Solution by K. Satyanarayana.

From the result (5·1) in the paper on 'Some results connected with triangles in perspective,' *Math. Student* Vol. II 49–58, it is clear that the loci of α, β, γ are got as follows:—

Let $\alpha_0 \beta_0 \gamma_0$ be a *particular* position of $\alpha\beta\gamma$; O_0 the centre of perspective; and A_0'', B_0'', C_0'' points so that $O_0 A_0'' = \alpha_0 A_0$ vectorially etc. Also let $\sum_0$, $\sum_0'$, $\sum_0''$ be the † s. s. p. members of the circum-conic families of $O_0 A_0 B_0 C_0$; $O_0 \alpha_0 \beta_0 \gamma_0$; $O_0 A_0'' B_0'' C_0''$. Let $\sum_0''$ become $\sum_\alpha$, $\sum_\beta$, $\sum_\gamma$ when displaced respectively by vectors $O_0\alpha_0$, $O_0\beta_0$, $O_0\gamma_0$. Then $\sum_\alpha$, $\sum_\beta$, $\sum_\gamma$ are the loci of α, β, γ.

If $H\alpha$, $H\beta$, $H\gamma$ be their centres and $\sum$ be the circumconic of $H_\alpha H_\beta H_\gamma$ s. s. p to $\sum_0$, it is clear that H_α, H_β, H_γ, $\sum$ and H (the centre of $\sum$) are fixed while $\alpha\beta\gamma$ moves as in the question.

† s. s. p. stands for 'similar and similarly placed.'

Also if H_0, H_0', H_0'' be the centres of $\sum_0$, $\sum_0'$, $\sum_0''$

$H_0' H = \alpha_0 H^{\alpha} = O_0 H_0''$ vectorially ... (1)

But by projecting $\sum_0$, $\sum_0'$, $\sum_0''$ into circles or otherwise it follows that $O_0 H_0'$ and $O_0 H_0''$ are respectively parallel to $H_0'' H_0$ and $H_0' H_0$ so that

$$H_0' H_0 = O_0 H_0'' \text{ vectorially} \quad \ldots \quad (2)$$

$\therefore$ H_0 is the same as H from (1) and (2).

So $\sum_0$ which is the locus of the centres of perspective [*vide* result (5·1) referred to above] corresponding to the triangle $A_0 B_0 C_0$ is concentric and s. s. p. with $\sum$ ... (3)

In the motion of $\alpha\beta\gamma$ contemplated in the question i.e. when the loci of α, β, γ are $\sum_\alpha$, $\sum_\beta$, $\sum_\gamma$ respectively, it follows that if ABC be *any* fixed triangle with which $\alpha\beta\gamma$ maintains perspectivity and if O be the centre of perspective of ABC and a particular position of $\alpha\beta\gamma$, the s. s. p. members of the circum-conic families of OABC, $O\alpha\beta\gamma$ should be s. s. p. to the loci of α, β, γ [*vide* result (5·1) above]. Hence O lies on the circum-conic of $\alpha\beta\gamma$ s. s. p. to $\sum$. Conversely if O be *any* point on the circum-conic of $\alpha\beta\gamma$ (s. s. p. to $\sum$) in *some* position of $\alpha\beta\gamma$, the fixed triangle ABC is got as follows:—

Let $\sum''$ be $\sum_\alpha$ displaced by vector αO. Let $O\alpha$, $O\beta$, $O\gamma$ cut $\sum''$ in A″, B″, C″. Then A, B, C are points so that OA″, OB″, OC″ are respectively equivalent (vectorially) to αA, βB, γC.

Thus there is an infinity of fixed triangles ABC and in each case, [as in (3)] the locus of the centre of perspective is concentric and s. s. p. with $\sum$.

Question 1745

(D. R. KAPREKAR):—We have

$$91 \times 819 = 74529$$

and also

$$\underline{9}9\underline{0}1 \times \underline{9}8\underline{0}1\underline{9}9 = \underline{9}7\underline{0}4\underline{9}5\underline{0}2\underline{9}9$$

where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. (The inserted figures are shown underlined). It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on.

It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

Solution by Johannes Mahrenholz, Cottbus, Germany.

It is

$$\{10^n(10^n - 1) + 1\} \times \{10^n(10^{2n} - 2\cdot 10^n + 2) - 1\}$$
$$= (10^{2n} - 10^n + 1)\cdot(10^{3n} - 2\cdot 10^{2n} + 2\cdot 10^n - 1)$$
$$= (10^{5n} - 3\cdot 10^{4n} + 5\cdot 10^{3n} - 5\cdot 10^{2n} + 3\cdot 10^n - 1).$$

Putting $n = 1$ and $n = 2$ we have the result given in the question.

Other examples are :—

$3\cdot 3 = 9$	$6\cdot 6 = 36$
$33\cdot 33 = 1089$	$66\cdot 66 = 4356$
$333\cdot 333 = 110889$	$666\cdot 666 = 443556$
$9\cdot 7 = 63$	$9\cdot 8 = 72$
$99\cdot 77 = 7623$	$99\cdot 88 = 8712$
$999\cdot 777 = 776223$	$999\cdot 888 = 887112$
$9\cdot 9 = 81$	$11\cdot 11 = 121$
$99\cdot 99 = 9801$	$101\cdot 101 = 10201$
$999\cdot 999 = 998001$	$1001\cdot 1001 = 1002001$

Question 1747

(K. RANGASWAMI) :—Solve the following equations for u, v, w:

$$u(m-k)^2 + vn^2 + wm^2 = \lambda p^2$$
$$un^2 + v(n-l)^2 + wl^2 = \lambda q^2$$
$$um^2 + vl^2 + w(l-m)^2 = \lambda r^2.$$

Solution by Johannes Mahrenholz, Cottbus, Germany.

The determinant of the equations is

$$2(nl - mn + ml)(mn + ml - nl)(mn - ml + nl).$$

Multiplying the three given equations in turn by the factors $u - mn + ml$, $mn + ml - nl$, $mn - ml + nl$ and adding we get

$$\lambda\{p^2(nl - mn + ml) + q^2(mn + ml - nl) + r^2(mn - ml + nl)\}$$
$$= 2\{m^2n^2 - l^2(m-n)^2\}u + 2\lambda p^2l^2$$
$$= 2\{n^2l^2 - m^2(n-l)^2\}v + 2\lambda q^2m^2$$
$$= 2\{m^2l^2 - n^2(l-m)^2\}w + 2\lambda r^2n^2.$$

$$u = \lambda\,\frac{p^2(nl - mn + ml - 2l^2) + q^2(mn + ml - nl) + r^2(mn - ml + nl)}{2\{m^2n^2 - l^2(m-n)^2\}};$$

$$v = \lambda\,\frac{p^2(nl - mn + ml) + q^2(mn + ml - nl - 2m^2) + r^2(mn - ml + nl)}{2\{n^2l^2 - m^2(n-l)^2\}};$$

$$w = \lambda\,\frac{p^2(nl - mn + ml) + q^2(mn + ml - nl) + r^2(mn + nl - ml - 2n^2)}{2\{m^2l^2 - n^2(l-m)^2\}}.$$

QUESTIONS FOR SOLUTION

1758. (D. R. KAPREKAR)—Prove without actual multiplication that the last nine digits of the huge number N given below are 999977778.

$$N = (546194538) \times \{1 + 24580 + (24580)^2 + \cdots\cdots + (24580)^{10}\}.$$

1759. (K. RANGASWAMI)—Show that if x is a positive number other than unity, and r and s are positive integers, then

$$\frac{x^r - x^{-r}}{r} \gtrless \frac{x^s - x^{-s}}{s}$$

according as $r >$ or $< s$.

1760. (A. A. KRISHNASWAMY AYYANGAR)—Show that for any real value of x, $\sqrt{\pi/2}\,(1 - e^{-x^2})$ gives a better numerical approximation to $\int_0^x e^{-\frac{1}{2}x^2}\,dx$ than $\sqrt{\pi/2}\,(1 - e^{-x^2/2})$

1761. (C. G. PENDSE)—Evaluate

$$\prod_{r=1}^{n-1} \tan\left(\theta + \frac{r\pi}{2n}\right)$$

in terms of θ and n.

1762. (R. RAMANUJAM)—If p, q, r represent the lengths of the altitudes of a triangle, find an expression for the area in terms of p, q, and r.

1763. (B. R. VENKATARAMAN)—If r and R are the in and circumradii of any tetrahedron, prove that $r < R/2$.

1764. (R. V. BADAMI)—If $\theta_1 = \text{const.}$ $\theta_2 = \text{const.}$ $\cdots$ $\theta_n = \text{const.}$ are a set of integrals of the equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots\cdots = \frac{dx_n}{X_n} = \frac{dx}{X} \qquad \ldots \quad (1)$$

prove by using Appell's Theorem * that $\dfrac{1}{X}\,\dfrac{\partial\,(\theta_1 \cdots \theta_n)}{\partial\,(x_1 \cdots x_n)}$ is a last multiplier of the system (1)

* Appell's Theorem: If M is a last multiplier of (1), then MD is a last multiplier of the transformed system

$$\frac{dy_1}{y_1} = \frac{dy_2}{y_2} \quad \cdots \quad = \frac{dy_n}{y_n}$$

where $D = \partial(x_1 \cdots\cdots x_n\, x)/\partial(y_1 \cdots\cdots y_n y)$.

THE

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